The Navier-Stokes Equations with Low-Regularity Data in Weighted Function Spaces

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Zusammenfassung

Wir betrachten die Navier-Stokes Gleichungen in einem beschränkten Teilgebiet des $\mathbb{R}^n$ und einem Zeitintervall $(0, T)$. Dabei ist es unser Ziel, eine Lösungstheorie zu entwickeln, die eine möglichst geringe Regularität der Daten voraussetzt. Das bedeutet gleichzeitig, dass man eine große Lösungsklasse erhält, in welcher die Lösungen a priori keinerlei schwache Ableitungen besitzen. Dies wiederum macht es nötig, einen neuen Lösungsbegriff einzuführen, der allgemeiner ist als der der schwachen Lösung, die sogenannte sehr schwache Lösung.

Dieses Problem untersuchen wir in im Ort gewichteten Funktionenräumen. Das heißt, Daten und Lösungen sind enthalten in Lebesgue-, Sobolev- und Besselpotentialräumen, wobei jeweils bezüglich des Maßes $w\,dx$ integriert wird. Das Gewicht w ist dabei in der Klasse der Muckenhoupt Gewichte enthalten. Diese besteht gerade aus den nichtnegativen, lokal integrierbaren Gewichtsfunktionen w, für die der Maximaloperator

$$M : L^q_w(\mathbb{R}^n) \to L^q_w(\mathbb{R}^n)$$

stetig ist.

Als Vorbereitung wird die Lösbarkeit der Laplacegleichung $\Delta u = f$ sowie der Divergenzgleichung $\operatorname{div} u = k$ in gewichteten Räumen bewiesen. Desweiteren wird ein linearer Fortsetzungsoperator konstruiert, der einem Vektor $(g_1, ..., g_k)$ von auf dem Rand definierten Funktionen eine Funktion u zuordnet, die auf dem Gebiet definiert ist und für deren Normalenableitungen gilt $\frac{\partial^j}{\partial N^j} u|_{\partial\Omega} = g_j$, $j = 1, ..., k$.

Als nächstes beschäftigen wir uns mit den linearen Stokes Gleichungen. Im stationären wie im instationären Fall erhält man die Lösbarkeit zu den allgemeinsten Daten, die hier betrachtet werden, durch Dualisierung der starken Lösungen. Diese Lösungen weisen jedoch im allgemeinen so wenig Regularität auf, dass ihre Einschränkung auf den Rand nicht mehr wohldefiniert ist. Wohldefinierte Randbedingungen erfordern eine Einschränkung auf solche Daten, die sich in eine Distribution auf dem Gebiet und eine Distribution auf dem Rand des Gebiets zerlegen lassen. In diesem Kontext lassen sich Spuren noch immer nicht im klassischen Sinne verstehen, der klassische Spuroperator läßt sich jedoch auf einen geeigneten Banachraum fortsetzen, der alle sehr schwachen Lösungen bezüglich der beschriebenen Daten enthält.

Mit Hilfe von komplexer Interpolation zwischen der sehr schwachen und der starken Lösung wird die Lösungstheorie der stationären und instationären Stokes Gleichungen auf gewichtete Besselpotentialräume übertragen. Dies setzt eine Charakterisierung der Interpolationsräume der Lösungsräume sowie der Räume der Daten voraus.

Im instationären Fall führt die höhere Regularität der Besselpotentialräume zu neuen Problemen, wenn man Lösungen zu inhomogenen Randbedingungen und Divergenzen sucht. Insbesondere wird es notwendig, entsprechend der Ortsregularität auch eine höhere Zeitregularität der Randbedingung und der Divergenz zu fordern. Die verwen-

deten Methoden sind Halbguppentheorie, operatorwertige Fouriermultiplikatoren und wiederum Interpolationstheorie.

Schließlich wenden wir uns den nichtlinearen Navier-Stokes Gleichungen zu. Sowohl im stationären als auch im instationären Fall erhalten wir Existenz und Eindeutigkeit der Lösungen für kleine Daten. Im instationären Fall kann diese Kleinheit der Daten auch durch eine Beschränkung auf ein kurzes Zeitintervall realisiert werden.

Summary

We consider the Navier-Stokes equations in a bounded domain $\Omega \subset \mathbb{R}^n$ and a time interval $(0,T)$. It is our aim to develop a solution theory which requires a regularity of the data that is as low as possible. This means at the same time that one obtains a class of solutions that is so large that the solutions possess a priori no weak derivatives. This in turn makes it necessary to introduce a notion of solutions that is more general than the one of weak solutions, the so-called very weak solutions.

We study this problem in weighted function spaces. This means that data and solutions are taken from Lebesgue-, Sobolev- and Bessel Potential spaces, where we integrate with respect to the measure $w\,dx$. The weight function w is contained in the class of Muckenhoupt weights. This class consists of all non-negative and locally integrable weight functions w, for which the maximal operator

$$M : L^q_w(\mathbb{R}^n) \to L^q_w(\mathbb{R}^n)$$

is continuous.

As a preparation we study the Laplace equation $\Delta u = f$ as well as the divergence equation $\operatorname{div} u = k$ in weighted function spaces. Moreover we construct a linear extension operator, which maps a vector $(g_1, ..., g_k)$ of functions defined on the boundary $\partial\Omega$ to a function u, that is defined on the domain Ω and whose normal derivatives fulfill $\frac{\partial^j}{\partial N^j} u|_{\partial\Omega} = g_j$, $j = 1, ..., k$.

Next we investigate the linearized Stokes equations. In the stationary as well as in the instationary case one obtains the solvability with respect to the most general data that are considered in the work, by dualization of strong solutions. However, these solutions in general do not possess enough regularity to make their restriction to the boundary well-defined. Boundary values are meaningful only after a restriction to data that can be decomposed to a distribution on the domain and a distribution on the boundary. Still in this context the traces cannot be understood in the classical way but the classical trace operator can be extended continuously to an appropriate Banach space that contains all very weak solutions to the data described above.

With the help of complex interpolation between the very weak and the strong solutions the solution theory of stationary and instationary Stokes equations can be extended to weighted Bessel Potential spaces. This in turn requires a characterization of the interpolation spaces of the corresponding spaces of data and solutions.

In the instationary case the higher regularity of Bessel Potential spaces leads to some new difficulties, if one looks for solutions to inhomogeneous boundary conditions and divergences. In particular it is necessary to demand some higher time regularity of the data that corresponds to the space-regularity we want to prove for the solution. The methods we use are semi-group theory, operator-valued Fourier multipliers and again interpolation theory.

Finally we examine the nonlinear Navier-Stokes equations. In the stationary as in the instationary case one obtains existence and uniqueness of solutions for small data. In the instationary case this smallness can be realized by restricting the problem to a short time interval.

Contents

1 Introduction

We consider the *Navier-Stokes equations* with fully inhomogeneous data which are given by

$$\frac{\partial u}{\partial t} - \nu\Delta u + u \cdot \nabla u + \nabla p = f \quad \text{in } (0,T) \times \Omega, \tag{1.0.1}$$

$$\operatorname{div} u = k \quad \text{in } (0,T) \times \Omega, \tag{1.0.2}$$

$$u = g \quad \text{on } (0,T) \times \partial\Omega, \tag{1.0.3}$$

$$u(0) = u_0 \quad \text{in } \Omega. \tag{1.0.4}$$

They describe the motion of a viscous fluid in a domain $\Omega \subset \mathbb{R}^n$, where $n = 2$ and $n = 3$ are the physically relevant cases, and in a time interval $(0,T)$, where $T = \infty$ is possible. The viscosity ν of the fluid is assumed to be constant. For notational convenience we only consider the case $\nu = 1$. Then the general case can be obtained by a dilation and a coordinate transformation.

The unknowns are the velocity field u and the pressure p. The exterior force f, the divergence k, the boundary condition g, and the initial condition u_0 are the given data.

The Navier-Stokes equations have been studied by many mathematicians in the last century. Most results deal with weak and strong solutions, which means that one searches for solutions which are a priori at least once weakly differentiable in space.

One aim of this thesis is to enlarge the class of solutions to the Navier-Stokes equations. This includes that simultaneously we are choosing the data as general as possible. Since the solutions are a priori only integrable, i.e., we do not demand any differentiability properties of the velocity field u, an appropriate formulation of the problem is needed, the so-called *very weak solutions* to the Navier-Stokes equations. To come to this formulation one multiplies (1.0.1) with a sufficiently smooth test function ϕ with $\phi(t)|_{\partial\Omega} = 0$ and $\operatorname{div} \phi(t) = 0$ for every t and $\operatorname{supp} \phi \subset [0,T) \times \overline{\Omega}$. Then one applies formal integration by parts and obtains

$$\begin{aligned} &- \langle u, \phi_t \rangle_{\Omega,T} - \langle u, \Delta\phi \rangle_{\Omega,T} \\ &= \langle f, \phi \rangle_{\Omega,T} - \langle g, N \cdot \nabla\phi \rangle_{\partial\Omega,T} + \langle uu, \nabla\phi \rangle_{\Omega,T} + \langle ku, \phi \rangle_{\Omega,T} - \langle u_0, \phi(0) \rangle_{\Omega} \end{aligned} \tag{1.0.5}$$

using the identity $u \cdot \nabla u = \operatorname{div}(uu) - (\operatorname{div} u)u$. Applying the same procedure to (1.0.2) and a test function ψ, which does not necessarily vanish on the boundary, yields

$$-\langle u(t), \nabla\psi \rangle_{\Omega} = \langle k(t), \psi \rangle_{\Omega} - \langle g(t), N\psi \rangle_{\Omega} \tag{1.0.6}$$

for almost every t. Now, u is called a very weak solution to the Navier-Stokes equations if (1.0.5) and (1.0.6) are fulfilled for all test functions ϕ and ψ. Note that the information about the boundary values is preserved because $\nabla\phi$ and ψ do not necessarily vanish on the boundary. This or similar formulations have been introduced by Amann in [3], by

Amrouche and Girault in [5] and by Galdi, Simader and Sohr in [30]. In these articles as well as by Farwig, Galdi and Sohr in [17], [16], [18] and by Giga in [32] solvability with low-regularity data has been shown. In particular, the boundary conditions under consideration are contained in spaces of distributions on the boundary.

The theory of very weak solutions and the enlarged class of solutions lead to new uniqueness criteria for weak solutions that have been shown by Farwig, Kozono and Sohr in [19].

An important step in the analysis of the Navier-Stokes equations is the treatment of the linearized problem called *Stokes problem.* The instationary Stokes problem is obtained when replacing (1.0.1) by

$$\frac{\partial u}{\partial t} - \Delta u + \nabla p = f \quad \text{in} \quad (0,T) \times \Omega.$$

To obtain the stationary Stokes problem one replaces (1.0.1) by

$$-\Delta u + \nabla p = f \quad \text{in} \quad \Omega$$

and omits (1.0.4). In the linear case the regularity of the data can be chosen so low such that every $u \in L^r(0,T;L^q(\Omega))$ in the instationary case and every $u \in L^q(\Omega)$ in the stationary case is a very weak solution to the Stokes equations with respect to appropriate data. However, it is obvious that for such u the equation (1.0.3) is in general meaningless because restrictions to the boundary are not well-defined in $L^q(\Omega)$. Moreover, it turns out, that in this most general context boundary conditions are not needed to prove the unique solvability. In a second step, we restrict ourselves to more regular forces and divergences to ensure well-defined boundary values of our solution.

We investigate this problem in function spaces that are weighted in the space variable. More precisely, we consider Lebesgue, Sobolev, and Bessel potential spaces with respect to the measure $w\,dx$, where w is a weight function contained in the Muckenhoupt class A_q. This is the class of nonnegative and locally integrable weight functions, for which the expression

$$A_q(w) := \sup_Q \left(\frac{1}{|Q|} \int_Q w\,dx \right) \left(\frac{1}{|Q|} \int_Q w^{-\frac{1}{q-1}}\,dx \right)^{q-1}$$

is finite, where the supremum is taken over all cubes in $\mathbb{R}^n$.

One reason, why the class of Muckenhoupt weights is appropriate for analysis is that the maximal operator is continuous in weighted L^q-spaces, if and only if the weight function is a Muckenhoupt weight. Thus the powerful tools of harmonic analysis may be applied, cf. García-Cuerva and Rubio de Francia [31] and Stein [49].

Moreover, classical tools for the treatment of partial differential equations extend to function spaces with Muckenhoupt weights. As important examples we mention the multiplier theorems [31], [49], extension theorems of functions on a domain to functions on $\mathbb{R}^n$ shown by Chua [9], extension theorems of functions on the boundary to functions on the domain by Fröhlich [27] and embedding theorems by Fröhlich [28] using the continuity of singular integral operators by Sawyer and Wheeden [41]. A very important tool in the theory of the Navier-Stokes equations is the Helmholtz decomposition. It has been established for weighted spaces by Fröhlich in [24] and by Farwig in [15].

These tools were the base to treat the solvability of the Stokes and Navier-Stokes equations by Farwig and Sohr in [21] and by Fröhlich in [25], [26], [27], see also [38] and [37].

In addition, Muckenhoupt weights can even be helpful in the analysis of partial differential equations. One reason for this lies in the so-called extrapolation theorem [31], which states the following.

If $A : L^q_w(\mathbb{R}^n) \to L^q_w(\mathbb{R}^n)$ is a linear operator, which is continuous for one $q \in (1, \infty)$ and every Muckenhoupt weight $w \in A_q$ such that

$$\|A\|_{\mathcal{L}(L^q_w(\Omega))} \leq C,$$

with C depending only on $A_q(w)$ (see Definition 3.1.1 below), then A can be extended to a continuous operator $A : L^r_w(\mathbb{R}^n) \to L^r_w(\mathbb{R}^n)$ for every $r \in (1, \infty)$ and every $w \in A_r$ and the associated continuity continuity constant is bounded by C.

As an important consequence we obtain that if $\mathcal{T}$ is a uniformly bounded set of continuous linear operators on $L^q_w(\Omega)$ for every $w \in A_q$ and the bound is depending only on $A_q(w)$, it follows that $\mathcal{T}$ is R-bounded (see Definition 2.4.3 below). This R-boundedness can be used to prove continuity of operator-valued Fourier multipliers as shown by Weis in [54] and maximal regularity, cf. Denk, Hieber, Prüss [11].

As shown in [21] examples of Muckenhoupt weights are

$$\begin{array}{ll} w(x) = (1 + |x|)^\alpha, & -n < \alpha < n(q-1) \quad \text{or} \\ \operatorname{dist}(x, M)^\alpha, & -(n-k) < \alpha < (n-k)(q-1), \end{array}$$

where M is a compact k-dimensional Lipschitzian manifold. Thus, if one chooses a particular weight function, the developed theory can be used for a better control of the growth of the solution, for example for $|x| \to \infty$, in the neighborhood of a point or close to the boundary.

However, the weighted context also causes difficulties. As an example we mention that translations are in general not continuous. More precisely, for $f \in L^q_w(\Omega)$ the function f_h defined by $f_h(x) := f(x + h)$ is in general not contained in $L^q_w(\Omega)$, because $w(x)$ might be small, while $w(x - h)$ is not.

Another large difficulty concerns Sobolev-like embedding theorems. Such theorems are shown in [28] using the continuity of weakly singular integral operators treated in [41]. However, these theorems require strong assumptions to the weight function. In other words, compared to the unweighted case these embeddings cause a greater loss of regularity according to the Muckenhoupt class A_q in which the weight function is contained.

This problem becomes important when dealing with the Navier-Stokes equations, since on the one hand embedding theorems are crucial for estimating the nonlinear term and, on the other hand, we want to obtain results for the most general class of weight functions possible. It turns out that instead of restricting the class of weight functions it is possible to consider the problem of very weak solutions in spaces of higher regularity. This implies that different embeddings are required, which hold for a larger class of weight functions. As a rule, the more general the weight function is, the higher we have to choose the regularity of the data and the solution.

However, since it is our aim to keep the class of solutions as large as possible, we do not demand more regularity of data than necessary. To adapt this regularity smoothly to the weight function we consider the problem in weighted Bessel potential spaces.

This thesis is organized in the following way.

In Chapter 2 we collect some basic analytical tools needed in this work. We commence with a presentation of some definitions and theorems from the theory of analytic semigroups including fractional powers of the generator and maximal regularity. Moreover, since we are frequently dealing with Bessel potential spaces, we introduce some notations and properties of complex interpolation theory. In Section 2.4 some definitions and important theorems in the context of Banach space-valued function spaces and spaces of distributions are given. In particular, we concentrate on the continuity of Fourier multipliers.

In Chapter 3 we introduce Muckenhoupt weights and properties of them. In addition, we define and discuss weighted Lebesgue and Sobolev spaces.

For the treatment of the Navier-Stokes equations we need the solvability of the Laplace equation and a good understanding of the divergence operator and divergence-free functions in many places. Thus, in Chapter 4 we treat strong and very weak solutions to the Laplace equation in bounded domains. Moreover, we prove a weighted analogue to Bogowski's theorem. Finally, we present the Helmholtz decomposition in weighted spaces and in particular, a regularity result for the Helmholtz projection.

For the explanation of the boundary values in Section 6.3 one needs a sufficient amount of test functions. This, in turn, requires extension theorems in weighted spaces, which are derived in Chapter 5. More precisely, for given functions $g_1, ..., g_m$ on the boundary we find a function u defined on Ω whose k'th normal derivative is the function g_k. In the unweighted case, such theorems can be found in the book of Nečas [39], but note that, even when dealing with the unweighted case, the result that is presented here permits more general domains, i.e, one requires less regularity of the boundary. This is made possible by the choice of the charts according to Section 5.1.

Chapter 6 is devoted to the stationary Stokes resolvent problem. We start with the most general context in which every $u \in L^q_w(\Omega)$ appears as a very weak solution to the Stokes problem. Because of this generality, the definition of very weak solutions given in this thesis does not include any explicit boundary conditions. Instead, force and divergence carry the information of the boundary values since they are contained in spaces of functionals which do not consist of distributions on Ω. In the sequel, we prove higher regularity of the solution, provided the data is regular enough. In particular, it is shown in which sense the strong solutions are included in the theory of very weak solutions. In the last part of this chapter we present a low-regularity setting to define boundary conditions of the solution. This is only possible after a restriction of the data to forces and divergences, which can be decomposed into a functional that is supported by the boundary and that represents the boundary condition and a part, which is given by a distribution on Ω. This distribution represents the force or the divergence, respectively, as in the classical sense. We find a Banach space containing all solutions that correspond to such data and a continuous operator that acts on this Banach space and that coincides with the usual restriction for smooth functions.

To obtain better estimates for the nonlinear term, our solution theory of the Navier-

Stokes equations is developed in weighted Bessel potential spaces. Thus in Chapter 7 we introduce weighted Bessel potential spaces and present important properties. In particular, we are dealing with complex interpolation spaces of the spaces of data (or of solutions) corresponding to strong and to very weak solutions to the stationary Stokes equations. These interpolation spaces are characterized in terms of Bessel potential spaces on the domain Ω and on $\mathbb{R}^n$.

In Chapter 8 the theory of very weak solutions to the stationary Stokes equations is generalized to weighted Bessel potential spaces using complex interpolation. Moreover, we introduce a generalization of the Stokes operator in some Bessel potential spaces of negative order, which is appropriate in the context of very weak solutions. This operator generates a bounded analytic semigroup and can be used to deal with the instationary problem.

In Chapter 9 we consider the instationary Stokes equations. As in the stationary case, we start with the most general context in which every $u \in L^r(0,T;L^q_w(\Omega))$ is a very weak solution to the instationary Stokes equations corresponding to appropriate data. Here we obtain, by dualization of the strong solutions, the solvability concerning fully inhomogeneous data. However, when turning to more regular data the inhomogeneity of the boundary condition and the divergence causes new difficulties. In order to avoid these difficulties one has to adapt the time regularity of the data to the space regularity. We start with the component of the boundary condition which is tangential to the boundary and use R-boundedness and Banach space-valued Fourier multipliers to obtain the required estimates. Then the normal component of the boundary condition and the divergence can be realized by a gradient. The semigroup generated by the generalized Stokes operator helps us to establish a solution with the given initial condition. A solution to the given force is obtained by interpolation between the very weak and the strong solution. In the end, we put everything together, the part coming from the boundary condition and the divergence, the initial condition and the force.

In the Chapters 10 and 11 we turn to the nonlinear Navier-Stokes equations, where in Chapter 10 we treat the stationary case and in Chapter 11 the instationary case. We prove unique solvability for small data with the help of Banach's fixed point theorem. In the instationary case the smallness of the data can be realized by considering a short time interval. As a preparation we prove embedding theorems which are needed to estimate the nonlinear term. The solutions are constructed in Bessel potential spaces. In particular the solutions to the instationary problem are contained in

$$L^r(0,T;H^{\beta,q}_w(\Omega)) \hookrightarrow L^r(0,T;L^p(\Omega)),$$

where r,p fulfill Serrin's condition $\frac{2}{r}+\frac{n}{p} \leq 1$.

2 Preliminaries

2.1 Notation

Let $\mathbb{Z}$ be the set of all integers, $\mathbb{N}$ be the set of all positive integers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Moreover, as usual $\mathbb{C}$, $\mathbb{R}$ stands for the set of all complex or real numbers, respectively.

For a Borel set $B \subset \mathbb{R}^n$ we denote by $|B|$ the n-dimensional Lebesgue measure of B. If X is a Banach space, $r > 0$ and $x \in X$ then we denote by $B_r(x)$ the ball in X centered in x and with radius r. We omit x and write B_r, if the center is unimportant or known from the context.

For a Lipschitz domain Ω we denote by $\partial\Omega$ its boundary and by N the exterior normal vector.

For a multi-index $\alpha \in \mathbb{N}_0^n$ we write

$$|\alpha| = \sum_{i=1}^{n} \alpha_i \quad \text{and} \quad \partial^\alpha = \frac{\partial^{\alpha_1}}{x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{x_n^{\alpha_n}}.$$

For $x, y \in \mathbb{R}^n$ we write $x \cdot y$ for the usual scalar product in $\mathbb{R}^n$ and xy stands for the matrix $(x_i y_j)_{ij}$.

Let X be a Banach space and X' its dual space. Then for $x \in X$ and $x^* \in X'$ we write

$$\langle x^*, x\rangle := x^*(x).$$

If we want to emphasize that X is a space of functionals on some domain Ω or $\partial\Omega$ we write

$$\langle x^*, x\rangle_\Omega \quad \text{or} \quad \langle x^*, x\rangle_{\partial\Omega},$$

respectively. This includes the possibility that

$$\langle x^*, x\rangle_\Omega = \int_\Omega x^*(t)x(t)dt \quad \text{or} \quad \langle x^*, x\rangle_{\partial\Omega} = \int_{\partial\Omega} x^*(t)x(t)dt$$

if x and x^* are regular enough to be identified with functions. If we want to emphasize that we are dealing with functions and integrals, we write

$$(x^*, x)_\Omega = \langle x^*, x\rangle_\Omega \quad \text{or} \quad (x^*, x)_{\partial\Omega} = \langle x^*, x\rangle_{\partial\Omega}.$$

We will often deal with spaces of $\mathbb{R}^n$-valued functions and their dual spaces. Such a space can be seen as the Cartesian product $Y = X \times ... \times X$, with a Banach space X of scalar-valued functions. The dual space Y' can be identified with $X' \times ... \times X'$ via

$$\langle u, v\rangle := \sum_{i=1}^{n} \langle u_i, v_i\rangle = \sum_{i=1}^{n} u_i(v_i),$$

for $u \in Y'$, $v \in Y$ and $u_i \in X'$, $v_i \in X$, $i = 1, ..., n$.

As usual, for a measurable set Ω we denote by $L^1_{loc}(\Omega)$ the space of all locally integrable functions $f : \Omega \to \mathbb{R}^n$. We write $f \in L^q_{loc}(\Omega)$, if $|f|^q \in L^1_{loc}(\Omega)$.

Moreover, by $C^\infty(\overline{\Omega})$ we denote the space of all functions which are smooth in the interior of $\overline{\Omega}$ and such that every derivative extends to a continuous function on $\overline{\Omega}$. By $C_0^\infty(\Omega)$ we denote the space of smooth functions with compact support in Ω.

By $\mathcal{L}(X, Y)$ we denote the space of continuous linear operators from X to Y. Moreover, we set $\mathcal{L}(X) := \mathcal{L}(X, X)$.

2.2 Analytic Semigroups

We collect some important definitions and theorems in the context of analytic semigroups which can be found in [40] and [11] and which are needed in Chapters 8, 9 and 11.

Definition 2.2.1. Let X be a Banach space and let

$$\Delta := \{z \in \mathbb{C} \mid \phi_1 < \arg z < \phi_2\} \text{ for } \phi_1 < 0 < \phi_2, \quad |\phi_j| < \frac{\pi}{2}.$$

For $z \in \Delta$ let $T(z)$ be a bounded linear operator. The family $T(z)$, $z \in \Delta$ is an analytic semigroup in Δ if

1. $z \mapsto T(z)$ is analytic in Δ,
2. $T(0) = \mathrm{id}$ and $\lim_{z\to 0} T(z)x = x$ for every $x \in X$,
3. $T(z_1 + z_2) = T(z_1)T(z_2)$ for $z_1, z_2 \in \Delta$.

The linear operator A defined by

$$\mathcal{D}(A) = \left\{ x \in X \;\middle|\; Ax := \lim_{t\to 0, t>0} \frac{T(t)x - x}{t} \text{ exists} \right\}$$

is called the *infinitesimal generator* of the semigroup $T(t)$. Then $\mathcal{D}(A)$ is the domain of A. We write $\varrho(A)$ for the resolvent set of A.

For $\frac{\pi}{2} > \delta > 0$ we define

$$\Sigma_\delta := \left\{ \lambda \in \mathbb{C} \setminus \{0\} \;\middle|\; |\arg(\lambda)| < \delta + \frac{\pi}{2} \right\}$$

and

$$\Delta_\delta := \{\lambda \in \mathbb{C} \setminus \{0\} \mid |\arg(\lambda)| < \delta\}.$$

Theorem 2.2.2. *Let X be a Banach space and $A : \mathcal{D}(A) \subset X \to X$ a linear operator and $\varepsilon > 0$. Then the following assertions are equivalent.*

1. *A is the generator of a bounded analytic semigroup $\{T(t)\}_{t\in\Delta_\delta}$ for $0 < \delta < \varepsilon$.*
2. *The operator A is densely defined and closed. Moreover, $\varrho(A)$ contains the sector Σ_ε and for every $0 < \delta < \varepsilon$ there exists a constant M_δ such that*

$$\|\lambda(A - \lambda)^{-1}\|_{\mathcal{L}(X)} \le M_\delta \quad \textit{for every} \quad \lambda \in \Sigma_\delta.$$

If A fulfills the assertions in Theorem 2.2.2, then the semigroup $T(t)$ generated by A is given by

$$T(t) = -\frac{1}{2\pi i}\int_\Gamma e^{\lambda t}(A-\lambda)^{-1}d\lambda \quad \text{for every} \quad t > 0,$$

where $\Gamma \subset \Sigma_\varepsilon$ is a curve from $\infty e^{-i\sigma}$ to $\infty e^{i\sigma}$ for some $\frac{\pi}{2} < \sigma < \frac{\pi}{2} + \varepsilon$.

If $-A$ is the generator of an analytic semigroup with $0 \in \varrho(A)$, then the fractional powers A^α are well-defined for every $\alpha \in \mathbb{R}$. More precisely, for $\alpha > 0$ one sets

$$A^{-\alpha} := \frac{1}{2\pi i}\int_\Gamma (-\lambda)^{-\alpha}(A+\lambda)^{-1}d\lambda,$$

where $\Gamma \subset \Sigma_\varepsilon \supset \varrho(A)$ is a curve running from $\infty e^{-i\theta}$ to $\infty e^{i\theta}$ with $\theta \in (0, \frac{\pi}{2} + \varepsilon)$ which does not intersect the positive real axis.

Then $A^{-\alpha} \in \mathcal{L}(X)$ and is injective. We also consider its inverse

$$A^\alpha : \mathcal{D}(A^\alpha) \subset X \to X,$$

where $\mathcal{D}(A^\alpha) = \{A^{-\alpha}y \mid y \in X\}$ and $A^\alpha x = (A^{-\alpha})^{-1}x$ for every $x \in \mathcal{D}(A^\alpha)$. Then A^α is densely defined, injective and surjective.

Finally, we set $A^0 = \mathrm{id}$.

Then the fractional powers have the following properties.

- For $\alpha \in \mathbb{N}$ one has $A^\alpha = A \underbrace{\circ \ldots \circ}_{\alpha\times} A$.
- One has $A^\alpha A^\beta x = A^{\alpha+\beta}x$ for every $\alpha, \beta \in \mathbb{R}$ and every $x \in \mathcal{D}(A^\gamma)$, where $\gamma = \max\{\alpha, \beta, \alpha + \beta\}$.

Theorem 2.2.3. *Let A be the infinitesimal generator of an analytic semigroup $T(t)$ on a Banach space X and let 0 be contained in the resolvent set of A. Then one has for every $\alpha \geq 0$:*

1. *$T(t) : X \to \mathcal{D}(A^\alpha)$.*
2. *For every $x \in \mathcal{D}(A^\alpha)$ one has $T(t)A^\alpha x = A^\alpha T(t)x$.*
3. *For every $t > 0$ the operator $A^\alpha T(t)$ is bounded and there exists $\delta > 0$ and a constant $M_\alpha = M_\alpha(\delta)$ such that*

$$\|A^\alpha T(t)\|_{\mathcal{L}(X)} = M_\alpha t^{-\alpha}e^{-\delta t}.$$

Let A be the generator of a bounded analytic semigroup. We consider the Cauchy problem

$$u'(t) + Au(t) = f(t), \quad t > 0, \quad u(0) = 0.$$

For a given $f \in L^r(\mathbb{R}_+; X)$ the *mild solution* u is given by the Variation of Constants formula

$$u(t) = \int_0^t T(t-s)f(s)ds.$$

We say that A has *maximal regularity*, if for each $f \in L^r(\mathbb{R}_+; X)$ the mild solution $u(t)$ is weakly differentiable, takes values in $\mathcal{D}(A)$, $Au \in L^r(\mathbb{R}_+; X)$ and fulfills the estimate

$$\|u'\|_{L^r(\mathbb{R}_+;X)} + \|Au\|_{L^r(\mathbb{R}_+;X)} \leq c\|f\|_{L^r(\mathbb{R}_+;X)}.$$

2.3 Complex Interpolation Theory

We sum up some facts and notation about complex interpolation theory we will frequently need in this text, in particular in Chapter 8.1.

Let X_1, X_2 be two Banach spaces continuously embedded into a common topological vector space. Then $\{X_1, X_2\}$ is called an interpolation couple. Furthermore, let

$$D = \{z \in \mathbb{C} \mid 0 < \operatorname{Re} z < 1\}.$$

We define $F(X_1, X_2)$ to be the space of all bounded and holomorphic functions f from D to $X_1 + X_2$ which are extendable to continuous functions on $\overline{D}$ such that $f(j + yi)$ is continuous on $\mathbb{R}$ with values in X_{j+1}, $j = 0, 1$, and

$$\sup\{\|f(iy)\|_{X_1} \mid y \in \mathbb{R}\} < \infty \quad \text{and} \quad \sup\{\|f(iy+1)\|_{X_2} \mid y \in \mathbb{R}\} < \infty.$$

Then $F(X_1, X_2)$ is a Banach space equipped with the norm

$$\|f\|_{F(X_1,X_2)} = \max\left\{\sup_{y\in\mathbb{R}} \|f(iy)\|_{X_1}, \sup_{y\in\mathbb{R}} \|f(iy+1)\|_{X_2}\right\}.$$

Now for $0 < \theta < 1$ one defines the complex interpolation space by

$$[X_1, X_2]_\theta = \{f(\theta) \mid f \in F(X_1, X_2)\},$$

equipped with the norm

$$\|x\|_{[X_1,X_2]_\theta} = \inf\{\|f\|_{F(X_1,X_2)} \mid f \in F(X_1, X_2) \text{ and } f(\theta) = x\}.$$

Theorem 2.3.1. *Let $0 < \theta < 1$ and $X_1 \subset X_2$ with continuous and dense embedding. Then one has*

1. *X_1 is densely and continuously embedded into $[X_1, X_2]_\theta$.*
2. *(Reiteration)*
$$\big[[X_1, X_2]_\lambda, [X_1, X_2]_\mu\big]_\theta = [X_1, X_2]_\eta$$
where $\lambda, \mu \in [0, 1]$ and $\eta = (1-\theta)\lambda + \theta\mu$.
3. *(Duality) Let X_1 and X_2 be reflexive. Then*
$$[X_1, X_2]'_\theta = [X'_1, X'_2]_\theta.$$
4. *Let $\{Y_1, Y_2\}$ be another interpolation couple with $Y_1 \subset Y_2$. Moreover let $T : X_i \to Y_i$ be a continuous linear operator for $i = 1, 2$. Then*
$$T : [X_1, X_2]_\theta \to [Y_1, Y_2]_\theta$$
is continuous with operator norm bounded by $\|T\|^{1-\theta}_{\mathcal{L}(X_1,Y_1)} \|T\|^{\theta}_{\mathcal{L}(X_2,Y_2)}$.

5. *Let* $\{X_1, X_2\}$ *and* $\{Y_1, Y_2\}$ *interpolation couples such that* $\{X_1, X_2\}$ *is a retract of* $\{Y_1, Y_2\}$, *i.e., there exist continuous linear operators*

$$I : X_1 + X_2 \to Y_1 + Y_2 \quad \textit{and} \quad P : Y_1 + Y_2 \to X_1 + X_2,$$

such that $PI = \mathrm{id}_{X_1+X_2}$ *and*

$$I : X_i \to Y_i \quad \textit{and} \quad P : Y_i \to X_i, \quad i = 1, 2$$

are continuous. Then $[X_1, X_2]_\theta = P[Y_1, Y_2]_\theta$ *for* $\theta \in [0, 1]$*. The norms*

$$\|u\|_{[X_1,X_2]_\theta} \quad \textit{and} \quad \inf\{\|U\|_{[Y_1,Y_2]_\theta} \mid PU = u\}$$

are equivalent and the equivalence constants depend only on the continuity constants of I *and* P.

Proof. All assertions can be found in [52] or [6]. For the assertions on the constants in 5. we remark that for $u \in [X_1, X_2]_\theta$ one has by 3.

$$\|u\|_{[X_1,X_2]_\theta} = \inf_{PU=u} \|PU\|_{[X_1,X_2]_\theta} \le c_1 \inf_{PU=u} \|U\|_{[Y_1,Y_2]_\theta} \le c_1\|Iu\|_{[Y_1,Y_2]_\theta} \le c_2\|u\|_{[X_1,X_2]_\theta}.$$

□

An operator P as in Theorem 2.3.1.4. is called retraction, the associated operator I is called coretraction (belonging to P).

2.4 Banach Space-Valued Functions and Distributions

In this section we collect some basic definitions and properties of spaces of functions and distributions with values in a Banach space.

Let X be a Banach space and $I \subset \mathbb{R}$ an open interval.

By $C_0^\infty(I; X)$, we denote the space of compactly supported smooth functions with values in X. The space of X-valued distributions is

$$\mathcal{D}'(I; X) := \mathcal{L}(C_0^\infty(I; \mathbb{R}), X).$$

By [4, Corollary 1.4.10] and [42, IV 9.9] one has

$$\mathcal{D}'(I; X) \cong (C_0^\infty(I; X'))'.$$

For $1 \le r < \infty$ we denote the space of strongly Lebesgue measurable functions cf. [55],

$$u : I \to X \ \text{ such that } \ \|u\|_{L^r(I;X)} := \left(\int_I \|u(t)\|_X^r dt\right)^{\frac{1}{r}} < \infty$$

by $L^r(I; X)$. Then, assuming in addition that X is reflexive, one has by [12], II Corollary 13 and IV Theorem 1,

$$(L^r(I; X))' = L^{r'}(I; X') \quad \text{for} \quad 1 < r < \infty. \tag{2.4.1}$$

The latter equality holds whenever X has the Radon-Nikodym property, cf. [12]. In particular, every reflexive Banach space has this property.

Moreover, for $u \in L^r(I;X)$ we define the distributional derivative $u_t = \partial_t u$ by

$$\partial_t u := \left[C_0^\infty(I,\mathbb{R}) \ni \phi \mapsto -\int_I u \partial_t \phi \right] \in \mathcal{D}'(I;X).$$

As in the scalar-valued case we define the X-valued Sobolev space by

$$W^{k,r}(I;X) := \left\{ u \in L^r(I;X) \;\middle|\; \|u\|_{W^{k,r}(X)} := \sum_{j=0}^{k} \|\partial_t^j u\|_{L^r(X)} < \infty \right\}, \quad k \in \mathbb{N}_0,$$

and

$$W^{-k,r}(I;X) := \left(W_0^{k,r'}(I;X') \right)', \quad \text{where } \; W_0^{k,r'}(I;X') := \overline{C_0^\infty(I;X')}^{W^{k,r'}(I;X')}, \quad k \in \mathbb{N}.$$

By [2], Theorem III.1.2.2, every $u \in W^{1,r}(I,X)$ is absolutely continuous. In particular for every $t \in I$ the evaluation $u(t) \in X$ is well-defined. Moreover, for $u \in W^{1,r}(I;X)$ and $v \in W^{1,r'}(I;X')$ one has the integration by parts formula

$$\int_0^T \langle u_t(t), v(t)\rangle_{X,X'} dt = \langle u(T), v(T)\rangle_{X,X'} - \langle u(0), v(0)\rangle_{X,X'} - \int_0^T \langle u(t), v_t(t)\rangle_{X,X'} dt.$$

Let X be a Banach space and $\mathcal{S}(\mathbb{R};X)$ be the space of rapidly decreasing functions $f : \mathbb{R} \to X$ and

$$\mathcal{S}'(\mathbb{R};X) := \mathcal{L}(\mathcal{S}(\mathbb{R};\mathbb{R});X).$$

For $f \in L^1(\mathbb{R};X)$ we write

$$\mathcal{F}f(t) := \hat{f}(t) := \int_{\mathbb{R}} e^{-its} f(s) ds$$

for the Fourier transform and $\check{f} := \mathcal{F}^{-1} f$ for the inverse Fourier transform of f. For $f \in S'(\mathbb{R};X)$ we set

$$\mathcal{F}f(\phi) := \hat{f}(\phi) := f(\hat{\phi}) \;\text{ for every }\; \phi \in \mathcal{S}(\mathbb{R};\mathbb{R}).$$

For $f \in L^1(\mathbb{R};X) \subset S'(\mathbb{R};X)$ the two definitions coincide.

In the whole, the continuity of multiplier theorems will be of great importance. In the Banach space-valued context these theorems are proved for a subclass of reflexive spaces called UMD-spaces.

Definition 2.4.1. A Banach space X is called a UMD-space if the Hilbert transform,

$$Hf(x) = PV - \int_{\mathbb{R}} \frac{1}{t-s} f(s) ds, \quad f \in \mathcal{S}(\mathbb{R};X),$$

extends to a bounded linear operator on $L^p(\mathbb{R};X)$ for every $1 < p < \infty$.

We quote the following theorem from Zimmermann [56].

Theorem 2.4.2. *Let X be a UMD-space. Then for every $p \in (1, \infty)$ there is a constant $C < \infty$ such that for every bounded function $\psi : \mathbb{R}^n \to \mathbb{C}$ whose distributional derivatives $\partial^\gamma \psi$ of order $\gamma \leq (1, ..., 1)$ are represented on $\mathbb{R} \setminus \{0\}$ by bounded functions we have*

$$\|\mathcal{F}^{-1} \psi \mathcal{F} u\|_{L^p(\mathbb{R}^n;X)} \leq C \sup_{\xi \in \mathbb{R}^n \setminus \{0\}, \, \gamma \leq (1,...,1)} \left(|\xi|^{|\gamma|} |\partial^\gamma \psi(\xi)| \right) \|u\|_{L^p(\mathbb{R}^n;X)}.$$

Definition 2.4.3. Let X, Y be Banach spaces. A subset $\mathcal{T} \subset \mathcal{L}(X,Y)$ is called R-bounded if there is a constant $C > 0$ such that for all $T_1, ..., T_n \in \mathcal{T}$, $x_1, ..., x_n \in X$ and $n \in \mathbb{N}$ one has

$$\int_0^1 \left\| \sum_{j=1}^n r_j(u) T_j(x_j) \right\|_Y du \leq C \int_0^1 \left\| \sum_{j=1}^n r_j(u) x_j \right\|_X du,$$

where (r_j) is a sequence of independent, symmetric $\{1, -1\}$-valued random variables on $[0, 1]$, e.g. the Rademacher functions, cf. [44].

From this definition one obtains immediately the following lemma.

Lemma 2.4.4. *Let X, Y be Banach spaces.*

1. *Let $\mathcal{T} \subset \mathcal{L}(X,Y)$ be R-bounded and let W and Z be two further Banach spaces and $A \in \mathcal{L}(W,X)$ and $B \in \mathcal{L}(Y,Z)$. Then $B \circ T \circ A \subset \mathcal{L}(W,Z)$ is R-bounded.*

2. *Let $\mathcal{T}_1, \mathcal{T}_2 \subset \mathcal{L}(X,Y)$ be R-bounded. Then*

$$\mathcal{T}_1 + \mathcal{T}_2 = \{A + B \mid A \in \mathcal{T}_1, \; B \in \mathcal{T}_2\}.$$

The following theorem has been shown by Weis in [54, Theorem 3.4].

Theorem 2.4.5. *Let X and Y be UMD-spaces. Let*

$$\mathbb{R} \setminus \{0\} \ni t \mapsto M(t) \in \mathcal{L}(X,Y)$$

be a differentiable function such that the sets

$$\{M(t) \mid t \in \mathbb{R} \setminus \{0\}\} \quad \textit{and} \quad \{tM'(t) \mid t \in \mathbb{R} \setminus \{0\}\}$$

are R-bounded. Then $\mathcal{K} f = [M(\cdot)\hat{f}(\cdot)]^\vee$, $f \in C_0^\infty(\mathbb{R}, X)$, extends to a bounded linear operator

$$\mathcal{K} : L^r(\mathbb{R}; X) \to L^r(\mathbb{R}; Y) \quad \textit{for} \quad 1 < r < \infty.$$

3 Weighted Function Spaces

In this chapter we introduce the class of weight functions which is used throughout this thesis. In addition, we define weighted Lebesgue and Sobolev spaces and present their most important properties.

By a cube in $\mathbb{R}^n$ we mean the set $\prod_{j=1}^n I_j$ with the intervals $I_j = (x_j - r, x_j + r) \subset \mathbb{R}$, $j = 1, ..., n$, for $x \in \mathbb{R}^n$ and some $r > 0$.

3.1 Muckenhoupt Weights

Definition 3.1.1.

1. Let A_q, $1 < q < \infty$, the set of Muckenhoupt weights, be given by all $0 \leq w \in L^1_{loc}(\mathbb{R}^n)$ for which

$$A_q(w) := \sup_Q \left(\frac{1}{|Q|} \int_Q w \, dx \right) \left(\frac{1}{|Q|} \int_Q w^{-\frac{1}{q-1}} dx \right)^{q-1} < \infty. \tag{3.1.1}$$

   The supremum is taken over all cubes in $\mathbb{R}^n$. To avoid trivial cases, we exclude the case where w vanishes almost everywhere.

2. By A_1 we denote the set of all $0 \leq w \in L^1_{loc}(\mathbb{R}^n)$ such that there exists a constant $c > 0$ such that

$$\sup_{x \in Q} \frac{1}{|Q|} \int_Q w \, dy \leq c w(x) \quad \text{for almost every} \quad x \in \mathbb{R}^n, \tag{3.1.2}$$

   where the supremum is taken over all Q in $\mathbb{R}^n$ containing x. The infimum over all constants c, for which (3.1.2) is fulfilled for every $Q \subset \mathbb{R}^n$ is denoted by $A_1(w)$.

3. For $w \in A_q$ and a Borel set B we write

$$w(B) = \int_B w(x) dx.$$

4. A constant $C = C(w)$ is called A_q-consistent if for every $c_0 > 0$ it can be chosen uniformly for all $w \in A_q$ with $A_q(w) < c_0$, i.e. there exists $M = M(c_0)$ such that

$$C(w) < M \quad \text{for all} \quad w \in A_q \quad \text{with} \quad A_q(w) < c_0.$$

The A_q-consistence is of great importance since it is needed for the application of the Extrapolation Theorem [31, IV Lemma 5.18]. In particular this is used when showing the continuity of operator-valued Fourier multipliers and the maximal regularity of an operator; see [27] and [22] for details and applications. In this text this method is used in Section 9.4 in particular in Theorem 9.4.2.

We sum up some general properties of Muckenhoupt weights frequently used in this text.

Lemma 3.1.2. *1. Every $w \in A_q$, $q \geq 1$ defines a locally finite Borel measure and for $q > 1$ one has*

$$w(Q) \leq \left(\frac{|Q|}{|F|}\right)^q w(F)$$

for all cubes Q and all Borel sets $F \subset Q$ with $|F| > 0$.

In particular, we obtain that $w(F) = 0$ implies $|F| = 0$ and $w(\mathbb{R}^n) = \infty$.

2. *$A_q \subset A_p$ for $q < p$.*

3. *Let $w \in A_q$ for $q > 1$. Then there exists $s < q$ such that $w \in A_s$.*

4. *If $w \in A_1$, then w is locally bounded from below.*

Proof. 1. [49, V.1.7]

2. [31, IV Theorem 1.14]

3. [49, IX Prop. 4.5]

4. This is clear by definition.

□

Lemma 3.1.3. *Let $1 \leq q < \infty$, $w \in A_q$ and let $Q \subset \mathbb{R}^n$ be a cube. Then there exists a constant $c > 0$ such that*

$$|F|^q \leq cw(F) \quad \textit{for every Borel set} \quad F \subset Q.$$

Proof. For $q > 1$ see Lemma 3.1.2. Let $q = 1$. Then we obtain by definition

$$0 < d := \frac{w(Q)}{|Q|} \leq A_1(w) \left(\operatorname*{ess\,sup}_{x \in Q} \frac{1}{w(x)}\right)^{-1} = A_1(w) \operatorname*{ess\,inf}_{x \in Q} w(x).$$

Thus we obtain

$$w(F) = \int_F w(x)dx \geq \int_F \operatorname*{ess\,inf}_{x \in Q} w(x)dx = |F|(\operatorname*{ess\,inf}_{x \in Q} w(x)) \geq |F| \frac{d}{A_1(w)}.$$

Thus the assertion for $q = 1$ holds with $c = \frac{A_1(w)}{d}$. □

Lemma 3.1.4. *Let $1 < q < \infty$.*

1. *Let $w \in A_q$ and α be Lipschitz continuous with Lipschitz continuous inverse. Then $w_\alpha := w \circ \alpha \in A_q$ with $A_q(w_\alpha) \leq cA_q(w)$ with c independent of w.*

2. *For $w \in A_q$ let*

$$w^*(x', x_n) := \begin{cases} w(x', x_n) & \textit{for} \quad x_n \geq 0 \\ w(x', -x_n) & \textit{for} \quad x_n < 0. \end{cases} \tag{3.1.3}$$

Then $w^* \in A_q$ with $A_q(w^*) \leq cA_q(w)$ with c independent of w.

3. Let $w_1, w_2 \in A_q$. Then $\min(w_1, w_2) \in A_q$ and $\max(w_1, w_2) \in A_q$ with

$$A_q(\min(w_1, w_2)) \le c(q)(A_q(w_1) + A_q(w_2)) \quad \textit{and}$$
$$A_q(\max(w_1, w_2)) \le c(q)(A_q(w_1) + A_q(w_2)),$$

with $c(q)$ depending on q but not on w_1 and w_2.

Proof. 1., 2. [25, Lemma 1.2].
3. One has $\min(w_1, w_2)^{-\frac{1}{q-1}} = \max(w_1^{-\frac{1}{q-1}}, w_2^{-\frac{1}{q-1}}) \le w_1^{-\frac{1}{q-1}} + w_2^{-\frac{1}{q-1}}$. Thus we can calculate for every cube $Q \subset \mathbb{R}^n$

$$\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q \min(w_1, w_2)dx\right)\left(\frac{1}{|Q|}\int_Q \min(w_1, w_2)^{-\frac{1}{q-1}}dx\right)^{q-1} \\
&\le \left(\frac{1}{|Q|}\int_Q \min(w_1, w_2)dx\right)\left(\frac{1}{|Q|}\int_Q w_1^{-\frac{1}{q-1}}dx + \frac{1}{|Q|}\int_Q w_2^{-\frac{1}{q-1}}dx\right)^{q-1} \\
&\le 2^{q-1}\left[\left(\frac{1}{|Q|}\int_Q w_1 dx\right)\left(\frac{1}{|Q|}\int_Q w_1^{-\frac{1}{q-1}}dx\right)^{q-1}\right. \\
&\left. + \left(\frac{1}{|Q|}\int_Q w_2 dx\right)\left(\frac{1}{|Q|}\int_Q w_2^{-\frac{1}{q-1}}dx\right)^{q-1}\right] \\
&\le c(q)(A_q(w_1) + A_q(w_2)).
\end{aligned}$$

The assertion about the maximum can be proved analogously. □

3.2 Weighted Lebesgue Spaces

We introduce weighted Lebesgue spaces.

Definition 3.2.1. For $w \in A_q$ and an open set Ω we define

$$L^q_w(\Omega) := \left\{ f \in L^1_{loc}(\overline{\Omega}) \mid \|f\|_{q,w} := \left(\int_\Omega |f|^q w\, dx\right)^{\frac{1}{q}} < \infty \right\}.$$

For the classical Lebesgue space, i.e., the case $w = 1$ we write $L^q(\Omega)$.
It is easily seen that

$$(L^q_w(\Omega))' = L^{q'}_{w'}(\Omega) \quad \text{with} \quad \frac{1}{q} + \frac{1}{q'} = 1 \quad \text{and} \quad w' = w^{-\frac{1}{q-1}}. \tag{3.2.1}$$

Moreover, for every $q \in (1, \infty)$, $w \in A_q$ and a bounded measurable set Ω there exist $r_1, r_2 \in \mathbb{R}$ with $r_2 < q < r_1$ such that

$$L^{r_1}(\Omega) \hookrightarrow L^q_w(\Omega) \hookrightarrow L^{r_2}(\Omega). \tag{3.2.2}$$

This is shown in [25, Lemma1.3]. The second embedding can be rewritten more precisely as follows.

Lemma 3.2.2. *Let Ω be a bounded domain. If $1 \leq s$, $w \in A_s$ and $1 \leq p < \infty$, then for $q \geq sp$ one has*

$$L^q_w(\Omega) \hookrightarrow L^p(\Omega).$$

Note that by Lemma 3.1.2.3 for every $w \in A_q$, $q > 1$ there exists $s < q$ such that $w \in A_s$. Thus we can always choose $p > 1$ such that $q > sp$.

Proof. First we assume that $s > 1$. Since $\frac{q}{p} \geq s$ one has $w \in A_{\frac{q}{p}}$. Thus

$$w^{-\frac{1}{\frac{q}{p}-1}} \in A_{(\frac{q}{p})'} \subset L^1_{loc}(\overline{\Omega}) = L^1(\Omega).$$

Together with the Hölder inequality this yields

$$\int_\Omega |f|^p \, dx = \int_\Omega |f|^p w^{\frac{p}{q}} w^{-\frac{p}{q}} \, dx \leq \|f\|^p_{q,w} \left(\int_\Omega w^{-\frac{1}{\frac{q}{p}-1}} \, dx \right)^{\frac{q-p}{q}} = c\|f\|^p_{q,w}$$

for every $f \in L^q_w(\Omega)$.

If $s = 1$, then by Lemma 3.1.2.4 one can assume that w is bounded from below on Ω. This implies $L^q_w(\Omega) \hookrightarrow L^p_w(\Omega) \hookrightarrow L^p(\Omega)$. □

For a locally integrable function f we define the maximal operator M by

$$(Mf)(x) = \sup_{r>0} \frac{1}{|B_r(0)|} \int_{|y| \leq r} |f(x-y)| dy.$$

One has the following close connection between the Muckenhoupt class A_q and the maximal operator.

Theorem 3.2.3. *Let $1 < q < \infty$ and $w \in A_q$. Then the maximal operator M is continuous on $L^q_w(\mathbb{R}^n)$. More precisely, there exists an A_q-consistent constant c such that*

$$\|Mf\|_{q,w} \leq c\|f\|_{q,w} \quad \text{for every } f \in L^q_w(\mathbb{R}^n).$$

Vice versa if μ is a nonnegative Borel measure and M is bounded on $L^q(\mathbb{R}^n, \mu)$, then μ is absolutely continuous and $d\mu = w\,dx$ for some $w \in A_q$.

Proof. See [31], Theorems 2.1 and 2.9. For the A_q-consistence of the constants one has to re-read the proof of [31], Theorem 2.9. The reverse inclusion can be found in [49, 2.2]. □

Theorem 3.2.4. (Hörmander-Michlin Multiplier Theorem with Weights) *Let $m \in C^n(\mathbb{R}^n \setminus \{0\})$ fulfill the property*

$$|\partial^\alpha m(\xi)| \leq K|\xi|^{-|\alpha|}, \quad \text{for every } \xi \in \mathbb{R}^n \setminus \{0\}, \quad |\alpha| = 0, 1, ..., n,$$

for some constant $K > 0$. Then T defined by

$$\widehat{Tf} = m\hat{f} \quad \text{for } f \in \mathcal{S}(\mathbb{R}^n, \mathbb{R})$$

extends to a continuous operator on $L^q_w(\Omega)$ for every $q \in (1, \infty)$ and $w \in A_q$.

More precisely there exists an A_q-consistent c such that

$$\|Tf\|_{q,w} \leq c\|f\|_{q,w}$$

for every $f \in L^q_w(\Omega)$.

Proof. This is an immediate consequence of [31], Theorem 3.9. The same proof can be used to show the A_q-consistence of the continuity constant. □

3.3 Weighted Sobolev Spaces

Definition 3.3.1. Let $1 \le q < \infty$, $w \in A_q$, $k \in \mathbb{N}_0$ and let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain.

1. Set
$$W_w^{k,q}(\Omega) = \left\{ u \in L_w^q(\Omega) \;\middle|\; \|u\|_{k,q,w} := \sum_{|\alpha|\le k} \|D^\alpha u\|_{q,w} < \infty \right\}.$$

2. Moreover we set
$$W_{w,0}^{k,q}(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{k,q,w}}.$$
The dual space of it is denoted by
$$W_w^{-k,q}(\Omega) := (W_{w',0}^{k,q'}(\Omega))',$$
where one has to replace q' by ∞ in the case $q = 1$.

3. Using this for $k > 0$ we set
$$W_{w,0}^{-k,q}(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{W_w^{-k,q}(\mathbb{R}^n)}}.$$

These spaces are called weighted Sobolev spaces.

Note that in the case $k > 0$ the space $W_{w,0}^{-k,q}(\Omega)$ does in general not consist of distributions on Ω but of distributions on $\mathbb{R}^n$ supported by $\overline{\Omega}$. See Lemma 3.3.4 below for an equivalent characterization of $W_{w,0}^{-k,q}(\Omega)$ for $k > 0$.

Extensions to the whole space are often needed in this text. The existence of a continuous extension operator has been shown by Chua [9] and is stated in the following Theorem.

Note that Chua's version covers a far more general class of domains called (ε,∞)-domains. However, since our treatment of the Navier-Stokes equation requires at least $C^{1,1}$-boundaries, we renounce to introduce this technical notation.

Theorem 3.3.2. *Let Ω be a bounded Lipschitz domain and $N \in \mathbb{N}$. Choose $p_i \in [1,\infty)$, $w_i \in A_{p_i}$ and $k_i \in \mathbb{N}_0$, $i = 1, ..., N$. Then there exists an extension operator*
$$E : \bigcap_{i=1}^N W_{w_i}^{k_i,p_i}(\Omega) \to \bigcap_{i=1}^N W_{w_i}^{k_i,p_i}(\mathbb{R}^n),$$
i.e., $Eu|_\Omega = u$ and
$$\|Eu\|_{W_{w_i}^{k_i,p_i}(\mathbb{R}^n)} \le c\|u\|_{W_{w_i}^{k_i,p_i}(\Omega)} \quad \text{for} \quad i = 1, ..., N$$
and for every $u \in \bigcap_{i=1}^N W_{w_i}^{k_i,p_i}(\Omega)$.

Proof. This is a special case of [9, Theorem 4.1]. There Chua proves extension theorems for the class of (ε,∞)-domains. By [35] this class includes Lipschitz domains. □

A domain Ω is called an extension domain, if for every $q > 1$, $w \in A_q$ and $k \in \mathbb{N}$ there exists a continuous extension operator

$$E : W_w^{j,q}(\Omega) \to W_w^{j,q}(\mathbb{R}^n), \quad \text{for} \quad j = 0, ..., k.$$

By Theorem 3.3.2 Lipschitz domains are extension domains.

Since for $k \geq 1$ one has $W_w^{k,q}(\Omega) \subset W_{loc}^{k,1}(\overline{\Omega})$, the restriction $u \mapsto u|_{\partial\Omega}$ is well-defined. Thus we may use the following definition of weighted function spaces defined on the boundary.

Definition 3.3.3. For $k \in \mathbb{N}$, $q \in (1, \infty)$ and $w \in A_q$ we set

$$T_w^{k,q}(\partial\Omega) := (W_w^{k,q}(\Omega))|_{\partial\Omega}$$

equipped with the norm $\|\cdot\|_{T_w^{k,q}} = \|\cdot\|_{T_w^{k,q}(\partial\Omega)}$ of the factor space, i.e.,

$$\|g\|_{T_w^{k,q}(\partial\Omega)} := \inf \left\{ \|u\|_{W_w^{k,q}(\Omega)} \;\middle|\; u \in W_w^{k,q}(\Omega) \text{ and } u|_{\partial\Omega} = g \right\}.$$

Moreover, we set $T_w^{0,q}(\partial\Omega) = (T_{w'}^{1,q'}(\partial\Omega))'$.

By [25], [27] and [9] the spaces $L_w^q(\Omega)$, $W_w^{k,q}(\Omega)$, $W_{w,0}^{k,q}(\Omega)$ and $T_w^{k,q}(\partial\Omega)$ are reflexive Banach spaces in which $C_0^\infty(\overline{\Omega})$, $(C_0^\infty(\Omega)$, $C^\infty(\overline{\Omega})|_{\partial\Omega}$, respectively) are dense.

Note that by Nečas [39], Chapitre 2, §5, in the unweighted case one has

$$T_1^{k,q}(\partial\Omega) = W^{k-\frac{1}{q},q}(\partial\Omega) \quad \text{for} \quad k \in \mathbb{N} \quad \text{and} \quad T_1^{0,q}(\partial\Omega) = W^{-\frac{1}{q},q}(\partial\Omega).$$

In particular, the space $T_w^{0,q}(\partial\Omega)$ does not consist of functions but of distributions on $\partial\Omega$.

Lemma 3.3.4. *Let Ω be a bounded Lipschitz domain. For $k \in \mathbb{Z}$, $q \in (1, \infty)$ and $w \in A_q$ one has*

$$W_{w,0}^{-k,q}(\Omega) = \left(W_{w'}^{k,q'}(\Omega) \right)'.$$

Proof. For $k = 0$ this follows from the density of $C_0^\infty(\Omega)$ in $L_w^q(\Omega)$ and (3.2.1).

If $k < 0$ it follows from the definition and the reflexivity of $W_{w,0}^{-k,q}(\Omega)$.

It remains to prove the case $k > 0$. By definition, $W_{w,0}^{-k,q}(\Omega)$ is a closed subspace of $W_w^{-k,q}(\mathbb{R}^n)$. Thus, for $u \in \left(W_{w,0}^{-k,q}(\Omega) \right)'$ there exists by the Hahn-Banach theorem a functional $U \in \left(W_w^{-k,q}(\mathbb{R}^n) \right)' = W_{w',0}^{k,q'}(\mathbb{R}^n) = W_{w'}^{k,q'}(\mathbb{R}^n)$ with $\|U\|_{W_{w'}^{k,q'}(\mathbb{R}^n)} = \|u\|_{(W_{w,0}^{-k,q}(\Omega))'}$ and

$$U|_{C_0^\infty(\Omega)} = u|_{C_0^\infty(\Omega)}.$$

The equality $W_{w',0}^{k,q'}(\mathbb{R}^n) = W_{w'}^{k,q'}(\mathbb{R}^n)$ holds by the density of $C_0^\infty(\Omega)$ in $W_{w'}^{k,q'}(\mathbb{R}^n)$. This means u can be identified with the function $U|_\Omega \in W_{w'}^{k,q'}(\Omega)$ which fulfills

$$\|U|_\Omega\|_{W_{w'}^{k,q'}(\Omega)} \leq c\|u\|_{(W_{w,0}^{-k,q}(\Omega))'}.$$

Vice versa let $u \in W^{k,q'}_{w'}(\Omega)$. Then by Theorem 3.3.2 there exists $U = Eu \in W^{k,q'}_{w'}(\mathbb{R}^n)$ with $U|_\Omega = u$ and we obtain by the continuity of E and the Hahn-Banach theorem

$$\begin{aligned} c\|u\|_{W^{k,q'}_{w'}(\Omega)} \geq & \|U\|_{W^{k,q'}_{w'}(\mathbb{R}^n)} \\ = & \sup_{\phi\in\mathcal{S},\|\phi\|_{W^{-k,q}_w(\mathbb{R}^n)}=1} |\langle U,\phi\rangle| \\ \geq & \sup_{\phi\in C_0^\infty(\Omega),\|\phi\|_{W^{-k,q}_w(\mathbb{R}^n)}=1} |\langle u,\phi\rangle| \\ = & \|u\|_{(W^{-k,q}_{w,0}(\Omega))'}. \end{aligned}$$

Thus we have shown $(W^{-k,q}_{w,0}(\Omega))' = W^{k,q'}_{w'}(\Omega)$. Now the reflexivity of the spaces proves the assertion. □

As in the unweighted case, one has the following connections between the function spaces on Ω and $\partial\Omega$ where Ω is a bounded Lipschitz domain, $q \in (1,\infty)$ and $w \in A_q$. Their proofs are clear by definition or can be found in [25].

- For $k \in \mathbb{N}$ the restriction

$$u \mapsto u|_{\partial\Omega}: \ W^{k,q}_w(\Omega) \to T^{k,q}_w(\partial\Omega)$$

  is continuous with continuity constant 1.

- For $u \in W^{1,q}_w(\Omega)$ and $v \in W^{1,q'}_{w'}(\Omega)$ one has the integration by parts formula

$$\int_\Omega u\partial_i v\,dx = \int_{\partial\Omega} uN_i v\,dS - \int_\Omega \partial_i u\,v\,dx,$$

  where S is the surface measure on $\partial\Omega$ and N_i is the ith component of the exterior normal vector N.

- $W^{1,q}_{w,0}(\Omega) = \{u \in W^{1,q}_w(\Omega) \,|\, u|_{\partial\Omega} = 0\}$.

- There exists a linear continuous extension operator

$$F : T^{1,q}_w(\partial\Omega) \to W^{1,q}_w(\Omega)$$

  with $Fg|_{\partial\Omega} = g$ for every $g \in T^{1,q}_w(\partial\Omega)$.

- For $\phi \in C^{k-1,1}(\Omega)$ the multiplication operator

$$M_\phi : W^{k,q}_w(\Omega) \to W^{k,q}_w(\Omega), \quad u \mapsto u\phi,$$

  is continuous.

As in the unweighted case the embeddings between Sobolev spaces on bounded domains are compact as stated in the following lemma.

Lemma 3.3.5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $k \in \mathbb{N}_0$. Then the embedding*

$$W_w^{k+1,q}(\Omega) \hookrightarrow W_w^{k,q}(\Omega)$$

is compact.

Proof. By [25] the assertion is true for $k = 0$. The more general assertion follows by mathematical induction. □

Lemma 3.3.6. *Let Ω and $\mathcal{O}$ be two domains in $\mathbb{R}^n$ and*

$$\alpha : \overline{\mathcal{O}} \to \overline{\Omega}$$

a $C^{k-1,1}$-diffeomorphism, $k \geq 1$.

1. *The operator*

$$T : u \mapsto u \circ \alpha : \quad W_w^{k,q}(\Omega) \to W_{w\circ\alpha}^{k,q}(\mathcal{O})$$

is continuous.

2. *The same is true for the operator*

$$S : g \mapsto g \circ \alpha : \quad T_w^{k,q}(\partial\Omega) \to T_{w\circ\alpha}^{k,q}(\partial\mathcal{O}).$$

3. *For $k \geq 2$ the operator T extends to a continuous linear operator*

$$T : W_w^{-k+1,q}(\Omega) \to W_{w\circ\alpha}^{-k+1,q}(\mathcal{O}).$$

The continuity constants of T and S depend on $k, q, \alpha, \mathcal{O}$ but not on the weight function w.

Proof. 1. The case $k = 1$ has been proved in [25] Lemma 3.17. Assume $\alpha \in C^{k-1,1}(\overline{\mathcal{O}})$ and the asserted continuity holds for k replaced by j, $j < k$. Then

$$\begin{aligned} \|\nabla(u \circ \alpha)\|_{j,q,w\circ\alpha,\mathcal{O}} &= \|((\nabla u) \circ \alpha) \cdot \nabla\alpha\|_{j,q,w\circ\alpha,\mathcal{O}} \\ &\leq c\|(\nabla u) \circ \alpha\|_{j,q,w\circ\alpha,\mathcal{O}} \leq c\|u\|_{j+1,q,w,\Omega}. \end{aligned}$$

Thus $Tu \in W_{w\circ\alpha}^{j+1,q}(\mathcal{O})$ with $\|(u \circ \alpha)\|_{j+1,q,w\circ\alpha,\mathcal{O}} \leq c\|u\|_{j+1,q,w,\Omega}$. Hence mathematical induction proves the assertion.
2.This statement follows from the continuity of T and the identity $S(g) = T(u)|_{\partial\mathcal{O}}$, where $u \in W_w^{k,q}(\Omega)$ is an appropriate extension of g.
3. For the proof of the third assertion take $u \in L_w^q(\Omega)$ and $\phi \in C_0^\infty(\mathcal{O})$. Then change of variables yields

$$\begin{aligned} \langle u \circ \alpha, \phi\rangle_{\mathcal{O}} &= \int_\Omega u(y)\phi(\alpha^{-1}(y))|\det \nabla\alpha^{-1}(y)|dy \\ &\leq \|u\|_{-k+1,q,w,\Omega}\|(\det \nabla\alpha^{-1})\phi \circ \alpha^{-1}\|_{k-1,q',w',\Omega} \\ &\leq c\|u\|_{-k+1,q,w,\Omega}\|\phi\|_{k-1,q',w'\circ\alpha,\mathcal{O}}. \end{aligned}$$

Since $w' \circ \alpha = (w \circ \alpha)'$, one obtains

$$\|u \circ \alpha\|_{-k+1,q,w\circ\alpha,\mathcal{O}} \leq c\|u\|_{-k+1,q,w,\Omega}.$$

By the density of the embedding $L_w^q(\Omega) \hookrightarrow W_w^{-k+1,q}(\Omega)$ the assertion is proved. □

For the treatment of the Navier-Stokes equation we often use the divergence-free version of the spaces defined above. Let

$$\begin{aligned} L^q_{w,\sigma}(\Omega) =& \{u \in L^q_w(\Omega) \mid \langle u, \nabla\phi\rangle_\Omega = 0 \text{ for all } \phi \in W^{1,q'}_{w'}(\Omega)\}, \\ W^{k,q}_{w,0,\sigma}(\Omega) :=& \{u \in W^{k,q}_{w,0}(\Omega) \mid \operatorname{div} u = 0\}, \end{aligned}$$

and let $C^\infty_{0,\sigma}(\Omega)$ be the space of smooth and divergence-free functions with compact support in Ω.

By [26] the following weighted analogue of the Poincaré inequality holds: there exists a constant $c = c(q, w) > 0$ such that

$$\begin{aligned} \|u\|_{q,w} \le c\|\nabla u\|_{q,w} \quad &\text{for every} \quad u \in W^{1,q}_w(\Omega) \quad \text{with} \quad \int_\Omega u = 0 \\ &\text{and for every} \quad u \in W^{1,q}_{w,0}(\Omega). \end{aligned} \tag{3.3.1}$$

In Section 9.4 we need A_q-consistence of the continuity constant of an extension operator which extends functions defined on a domain Ω to functions defined on $\mathbb{R}^n$. However, it is not easy to extract this from Chua's proof in [9]. We help ourselves with the following simpler and weaker result that guarantees extensions of functions defined on the half space $\mathbb{R}^n_+$ and for special weight functions, but with A_q-consistent constant.

Lemma 3.3.7. *Let $k \in \mathbb{N}$, $1 < q < \infty$ and $w \in A_q$. Then there exists a weight function $\tilde{w} \in A_q$ with $\tilde{w}|_{\mathbb{R}^n_+} = w|_{\mathbb{R}^n_+}$ and $A_q(\tilde{w}) \le cA_q(w)$, where c is independent of w and such that there exists a continuous linear extension operator*

$$E : W^{j,q}_w(\mathbb{R}^n_+) \to W^{j,q}_{\tilde{w}}(\mathbb{R}^n), \quad j = 0, ..., k,$$

such that

$$\|E\|_{\mathcal{L}(W^{j,q}_w(\mathbb{R}^n_+), W^{j,q}_{\tilde{w}}(\mathbb{R}^n))} \le c, \quad j = 0, ..., k,$$

with c independent of w.

Proof. We set

$$\tilde{w}(x', x_n) := \begin{cases} w(x', x_n) & \text{if } \ x_n > 0 \\ \min_{j=1,...,k+1} w(x', -jx_n) & \text{if } \ x_n < 0. \end{cases} \tag{3.3.2}$$

To estimate $A_q(\tilde{w})$ let

$$\bar{w}_j := \begin{cases} w(x', x_n) & \text{if } \ x_n > 0 \\ w(x', -jx_n) & \text{if } \ x_n < 0 \end{cases}$$

for $j = 1, ..., k + 1$. Moreover, let Q be a cube in $\mathbb{R}^n$. If $Q \subset \mathbb{R}^n_+$ then nothing is to prove.

We consider the case $Q \cap \mathbb{R}^n_- \neq \emptyset$. Let $Q^+ \subset \mathbb{R}^n_+$ and $Q^- \subset \mathbb{R}^n_-$ be two cubes of the same size as Q such that $Q \subset Q^+ \cup Q^-$. Then one has

$$\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q \bar{w}_j dx\right)\left(\frac{1}{|Q|}\int_Q \bar{w}_j^{-\frac{1}{q-1}} dx\right)^{q-1} \\
&\leq \frac{1}{|Q|^q}\left(\int_{Q^+} \bar{w}_j dx + j^{-1}\int_{jQ^+} \bar{w}_j(x', \frac{x_n}{-j})dx\right) \cdot \\
&\quad \left(\int_{Q^+} \bar{w}_j^{-\frac{1}{q-1}} dx + j^{-1}\int_{jQ^+} \bar{w}_j^{-\frac{1}{q-1}}(x', \frac{x_n}{-j})dx\right)^{q-1} \\
&\leq \left(\frac{j^n}{|jQ^+|}\left(j^{-1}+1\right)\int_{jQ^+} w dx\right) \cdot \left(\frac{j^n}{|jQ^+|}\left(j^{-1}+1\right)\int_{jQ^+} w^{-\frac{1}{q-1}} dx\right)^{q-1} \\
&\leq c(j,q)A_q(w),
\end{aligned}$$

where $jQ^+ \subset \mathbb{R}^n_+$ is a cube containing the cuboid $\{(x', jx_n) \mid x \in Q^+\}$. The case $Q \subset \mathbb{R}^n_-$ can be treated with similar arguments. This yields $A_q(\bar{w}_j) \leq c(j,q)A_q(w)$.

By Lemma 3.1.4 we thus obtain

$$A_q(\tilde{w}) = A_q(\min_j \bar{w}_j) \leq c\sum_j A_q(\bar{w}_j) \leq cA_q(w)$$

and c is independent of w.

Now for $u \in C^k(\overline{\mathbb{R}^n_+}) \cap W^{k,q}_w(\mathbb{R}^n_+)$ we define the extension as in the unweighted case [1] by

$$Eu(x) = \begin{cases} u(x) & \text{for } \ x_n > 0 \\ \sum_{j=1}^{k+1} \lambda_j u(x', -jx_n) & \text{for } \ x_n < 0, \end{cases}$$

where λ_j, $j = 1, ..., k+1$, is chosen such that $\sum_{j=1}^{k+1} \lambda_j (-j)^l = 1$ for $l = 0, ..., k$. Then $Eu \in C^k(\mathbb{R}^n)$ and one has the estimate

$$\|Eu\|_{W^{k,q}_{\tilde{w}}(\mathbb{R}^n)} \leq c\|u\|_{W^{k,q}_w(\mathbb{R}^n_+)}$$

using change of variables. $\square$

4 Laplacian and Divergence

A good understanding of the Laplacian and the divergence is crucial for the treatment of the Stokes and Navier-Stokes equations starting from Chapter 6. Thus, this chapter provides solvability results for the Laplace and Laplace resolvent equation in bounded domains and in the half space. In addition, we prove a weighted analogue to Bogowski's Theorem, i.e., it is shown that there exist solutions u to the equation $\operatorname{div} u = f$. Finally, we present the Helmholtz decomposition, including a regularity result.

4.1 Strong and Very Weak Solutions to the Laplace Equation

Recall that for $0 < \varepsilon \leq \frac{\pi}{2}$ the sector Σ_ε is defined by

$$\Sigma_\varepsilon := \left\{ \lambda \in \mathbb{C} \setminus \{0\} \mid \arg(\lambda) \leq \varepsilon + \frac{\pi}{2} \right\}.$$

The solvability of the resolvent problem of the Dirichlet-Laplacian in the whole space $\mathbb{R}^n$ and the half space $\mathbb{R}^n_+$ has been shown in [25]. More precisely one has:

Theorem 4.1.1. *Let $\Omega = \mathbb{R}^n$ or $\mathbb{R}^n_+$. Moreover let $0 < \varepsilon \leq \frac{\pi}{2}$, $1 < p < \infty$, $w \in A_q$. Then for every $\lambda \in \Sigma_\varepsilon$ and $f \in L^q_w(\Omega)$ there exists a unique solution $u \in W^{2,q}_w(\Omega)$ to*

$$(\lambda - \Delta)u = f \quad \textit{and, in the case} \;\; \Omega = \mathbb{R}^n_+, \quad u|_{\mathbb{R}^{n-1}} = 0.$$

It fulfills the estimate

$$\|\lambda u\|_{q,w} + \sqrt{|\lambda|}\|\nabla u\|_{q,w} + \|\nabla^2 u\|_{q,w} \leq c\|f\|_{q,w}$$

with $c = c(n, q, w, \varepsilon) > 0$, independent of $\lambda \in \Sigma_\varepsilon$.

For $f \in W^{-1,q}_w(\mathbb{R}^n_+)$ we call $u \in W^{1,q}_w(\mathbb{R}^n_+)$ a weak solution to the Laplace resolvent problem

$$(1 - \Delta)u = f \quad \text{and} \quad u|_{\mathbb{R}^{n-1}} = 0$$

in $\mathbb{R}^n_+$ if

$$u \in W^{1,q}_{w,0}(\mathbb{R}^n_+) \;\; \text{and} \;\; (\nabla u, \nabla\phi) = (f, \phi) \;\; \text{for every} \;\; \phi \in W^{1,q'}_{w',0}(\mathbb{R}^n_+).$$

The regularity assertion in the following Theorem is proved similarly to the classical unweighted case (see e.g. Evans [14], 6.3., Thm. 5).

Theorem 4.1.2. (Regularity of the Dirichlet Problem)
Let $1 < q < \infty$, $k \in \mathbb{Z}$, $k \geq -1$ and let $f \in W_w^{k,q}(\mathbb{R}^n_+)$. Then there exists a unique weak solution $u \in W_w^{k+2,q}(\mathbb{R}^n_+)$ to the Dirichlet Problem

$$(1-\Delta)u = f \quad \textit{and} \quad u|_{\mathbb{R}^{n-1}} = 0.$$

It fulfills the estimate $\|u\|_{k+2,q,w} \leq c\|f\|_{k,q,w}$, where $c = c(k,q,w)$.

The same is true for the solution u of $(1-\Delta)u = 0$, $u|_{\mathbb{R}^{n-1}} = g$, if $g \in T_w^{k+2,q}(\mathbb{R}^{n-1})$, i.e., it fulfills the estimate

$$\|u\|_{k+2,q,w} \leq c\|g\|_{T_w^{k+2,q}}.$$

Proof. For the existence of weak solutions in weighted spaces see [27]. This is the assertion for $k = -1$. From Theorem 4.1.1 we obtain the assertion for $k = 0$.

By mathematical induction we assume that $u \in W_w^{j,q}(\mathbb{R}^n_+)$, $2 \leq j < k+2$.
For $\alpha \in \mathbb{N}^n$ with $|\alpha| = j-1$ and $\alpha_n = 0$ we have

$$(1-\Delta)u^{(\alpha)} = f^{(\alpha)} \in L^q(\mathbb{R}^n_+), \qquad \text{with} \quad u^{(\alpha)}|_{\mathbb{R}^{n-1}} = 0,$$

where $u^{(\alpha)} := \frac{\partial^{|\alpha|}u}{\partial^{\alpha_1}\ldots\partial^{\alpha_n}}$ and $f^{(\alpha)} := \frac{\partial^{|\alpha|}f}{\partial^{\alpha_1}\ldots\partial^{\alpha_n}}$.

From the uniqueness of strong solutions in Theorem 4.1.1 it follows that $u^{(\alpha)} \in W_w^{2,q}(\mathbb{R}^n_+)$ with

$$\|u^{(\alpha)}\|_{2,q,w} \leq c\|f^{(\alpha)}\|_{q,w} \leq c\|f\|_{j-1,q,w}.$$

It follows that $u^{(\beta)} \in L_w^q(\mathbb{R}^n_+)$ for every $\beta \in \mathbb{N}^n$ with $|\beta| = j+1$ and $\beta_n = 0,1,2$.

We again apply mathematical induction to get rid of the restriction on β_n. Assume the assertion is proved for $\gamma \in \mathbb{N}^n$ with $|\gamma| = j+1$ and $\gamma_n \leq l$. If $\gamma_n = l+1$ we decompose $\gamma = \delta + (0,...,2)$ and obtain

$$u^{(\gamma)} = (\partial_{n,n}u)^{(\delta)} = \left(f - u - \sum_{\nu=1}^{n-1}\partial_{\nu,\nu}u\right)^{(\delta)} \in L_w^q(\mathbb{R}^n_+).$$

Hence $\|u\|_{j+1,q,w} \leq \|f\|_{k,q,w}$ and the induction is completed.

For the second assertion let $v \in W_w^{k+2,q}(\mathbb{R}^n_+)$ be an extension of g. Then we find a unique $u \in W_w^{k+2,q}(\mathbb{R}^n_+)$ with $(\mathrm{id}-\Delta)u = (\mathrm{id}-\Delta)v$ and $u|_{\mathbb{R}^{n-1}} = 0$. Thus $v-u$ solves the problem and by the first assertion it fulfills the estimate. □

Theorem 4.1.3. *Let $1 < q < \infty$, $w \in A_q$, $\lambda \in \Sigma_\varepsilon \cup \{0\}$ and let Ω be a bounded $C^{1,1}$-domain. Then for any $f \in L_w^q(\Omega)$ there exists a unique $u \in W_w^{2,q}(\Omega)$ solving*

$$(\lambda - \Delta)u = f \quad \textit{and} \quad u|_{\partial\Omega} = 0.$$

This function u fulfills the estimate

$$|\lambda|\|u\|_{q,w} + \|u\|_{2,q,w} \leq c\|f\|_{q,w},$$

where $c = c(\Omega, q, w, \lambda)$.

Proof. See Appendix A.1. □

In this thesis, we frequently use the space $Y^{2,q'}_{w'}(\Omega)$ and its dual space $Y^{-2,q}_{w}(\Omega) := \left(Y^{2,q'}_{w'}(\Omega)\right)'$, see Section 6.1 below for further details and applications.

Definition 4.1.4. Let $F \in Y^{-2,q}_{w}(\Omega)$. Then we call $u \in L^q_w(\Omega)$ a very weak solution to the Laplace equation $\Delta u = f$, if

$$\langle u, \Delta\phi\rangle = \langle F, \phi\rangle \tag{4.1.1}$$

holds for every $\phi \in Y^{2,q'}_{w'}(\Omega)$.

Theorem 4.1.5. *For every $f \in Y^{-2,q}_{w}(\Omega)$ there exists a unique very weak solution to the Laplace equation $\Delta u = f$. It fulfills*

$$\|u\|_{q,w} \le c\|f\|_{Y^{-2,q}_{w}(\Omega)}$$

with $c = c(q, w, \Omega) > 0$.

Proof. By Theorem 4.1.3 we know that the operator

$$\Delta : Y^{2,q'}_{w'}(\Omega) \to L^{q'}_{w'}(\Omega)$$

is invertible, we denote its inverse by Δ_D^{-1}. Thus we can define a functional u by $\langle u, v\rangle := \langle f, \Delta_D^{-1} v\rangle$ for every $v \in L^{q'}_{w'}(\Omega)$. Then

$$|\langle u, v\rangle| = |\langle f, \Delta_D^{-1} v\rangle| \le \|f\|_{Y^{-2,q}_{w}(\Omega)}\|\Delta_D^{-1} v\|_{2,q',w'} \le c\|f\|_{Y^{-2,q}_{w}(\Omega)}\|v\|_{q',w'}.$$

Thus $u \in (L^{q'}_{w'}(\Omega))' = L^q_w(\Omega)$ and $\|u\|_{q,w} \le c\|f\|_{Y^{-2,q}_{w}(\Omega)}$.

To show that u is a very weak solution to the Laplace equation we see that for any $\phi \in Y^{2,q'}_{w'}(\Omega)$

$$\langle u, \Delta\phi\rangle = \langle f, \Delta_D^{-1}\Delta\phi\rangle = \langle f, \phi\rangle.$$

Vice versa every very weak solution to the Laplace equation fulfills

$$\langle u, \phi\rangle = \langle u, \Delta\Delta_D^{-1}\phi\rangle = \langle f, \Delta_D^{-1}\phi\rangle.$$

This proves the uniqueness. $\square$

Corollary 4.1.6. *Let Ω be a bounded $C^{1,1}$-domain, $1 < q < \infty$ and $w \in A_q$. Then*

$$T^{0,q}_{w}(\partial\Omega) \cong \{u \in L^q_w(\Omega) \mid \Delta u = 0\},$$

i.e., these spaces are isomorphic.

Proof. By Theorem 5.2.2 below there exists a continuous linear operator

$$E : T^{1,q'}_{w'}(\partial\Omega) \to W^{2,q'}_{w'}(\Omega), \quad Eg|_{\partial\Omega} = 0 \;\text{ and }\; N \cdot \nabla Eg|_{\partial\Omega} = g.$$

Using this we define

$$\begin{aligned} B : \{u \in L^q_w(\Omega) \mid \Delta u = 0\} &\to T^{0,q}_{w}(\partial\Omega) \\ u &\mapsto [T^{1,q'}_{w'}(\partial\Omega) \ni \phi \mapsto \langle u, \Delta E\phi\rangle_\Omega]. \end{aligned}$$

It is easy to check that B is continuous. Vice versa, let

$$\begin{aligned} G : T_w^{0,q}(\partial\Omega) &\to \{u \in L_w^q(\Omega) \mid \Delta u = 0\} \\ g &\mapsto u, \end{aligned}$$

where u is the very weak solution to the Laplace equation $\Delta u = f$ with

$$F := [Y_{w'}^{2,q'}(\Omega) \ni \phi \mapsto \langle g, N \cdot \nabla\phi\rangle_{\partial\Omega}] \in Y_w^{-2,q}(\Omega).$$

Then one has by Theorem 4.1.5

$$\begin{aligned} \|Gg\|_{q,w} &\le c \sup_{\phi \in Y_{w'}^{2,q'}(\Omega), \|\phi\|_{2,q',w'}=1} |\langle g, N \cdot \nabla\phi\rangle_{\partial\Omega}| \\ &\le c \sup_{\phi \in Y_{w'}^{2,q'}(\Omega), \|\phi\|_{2,q',w'}=1} \|g\|_{T_w^{0,q}} \|N \cdot \nabla\phi\|_{T_{w'}^{1,q'}} \le c\|g\|_{T_w^{0,q}}. \end{aligned}$$

Moreover, since $F|_{C_0^\infty(\Omega)} = 0$, one has $\Delta u = 0$ in the sense of distributions.

Now it remains to check that $BG = \mathrm{id}$ and $GB = \mathrm{id}$. For $u \in \{v \in L_w^q(\Omega) \mid \Delta v = 0\}$ and $\phi \in Y_{w'}^{2,q'}(\Omega)$ one has

$$\begin{aligned} \langle GBu, \Delta\phi\rangle_\Omega &= \langle Bu, N \cdot \nabla\phi\rangle_{\partial\Omega} = \langle u, \Delta E(N \cdot \nabla\phi)\rangle_\Omega \\ &= \langle u, \Delta(E(N \cdot \nabla\phi) - \phi) + \Delta\phi\rangle_\Omega = \langle u, \Delta\phi\rangle_\Omega, \end{aligned}$$

where in the last equation we have used that by Corollary 5.2.3 below one has $E(N \cdot \nabla\phi) - \phi \in W_{w',0}^{2,q'}(\Omega)$ and that u is harmonic. By the uniqueness of very weak solutions we find $GBu = u$. Vice versa, for all $\zeta \in T_{w'}^{1,q'}(\partial\Omega)$

$$\langle BGg, \zeta\rangle_{\partial\Omega} = \langle Gg, \Delta E\zeta\rangle_\Omega = \langle g, N \cdot \nabla E\zeta\rangle_{\partial\Omega} = \langle g, \zeta\rangle_{\partial\Omega}.$$

Since ζ was arbitrary, we obtain $BGg = g$. □

4.2 A Property of Harmonic Functions

In unweighted spaces one can approximate a function $u \in L^q(\Omega)$ by functions $u_\lambda(x) := u(\lambda x)$, $\lambda \le 1$ and $\lambda \to 1$. Unfortunately in weighted function spaces this is not possible, since $u \in L_w^q(\Omega)$ does not imply $u_\lambda \in L_w^q(\Omega)$.

However, if the function u is harmonic, one has $u \in C^\infty(\Omega)$ which clearly implies $u_\lambda \in C^\infty(\overline{\Omega}) \subset L_w^q(\Omega)$. Moreover one can show that the above approximation is possible. This will be proved in the present section.

Lemma 4.2.1. *Let $\Omega \subset \mathbb{R}^n$ be strictly star-shaped, i.e., Ω is star-shaped with respect to every point of a ball $B_r(0)$, $r > 0$, with $\overline{B_r(0)} \subset \Omega$. Moreover, let $u \in L^q(\Omega)$ with $\Delta u = 0$. For $\lambda < 1$ we set $u_\lambda(x) := u(\lambda x)$. Then*

$$u_\lambda \xrightarrow{\lambda \to 1, \lambda < 1} u \quad \text{in } L_w^q(\Omega).$$

Proof. Let $d = \sup_{x\in\Omega} |x|$ and choose $K < \frac{r}{d}$. Then for every λ with $\frac{1}{2} < \lambda < 1$ one has

$$B_{K(1-\lambda)|x|}(\lambda x) \subset \Omega \quad \text{for every } x \in \Omega. \tag{4.2.1}$$

To show this let $y \in \mathbb{R}^n$, $|y - \lambda x| < K(1-\lambda)|x|$. For $z = \frac{y-\lambda x}{1-\lambda}$ we have

$$|z| \le \frac{(1-\lambda)\frac{r}{d}|x|}{1-\lambda} \le r.$$

Since Ω is star-shaped with respect to $z \in B_r(0)$, we have

$$y = \lambda x + (1-\lambda) z \in \Omega.$$

This proves (4.2.1). Now set

$$\tilde{u}(x) = \begin{cases} u(x) & \text{if } x \in \Omega \\ 0 & \text{if } x \in \mathbb{R}^n \setminus \Omega \end{cases} \in L^q_w(\mathbb{R}^n).$$

Take $x \in \Omega$ and $\lambda < 1$ fixed. Since u is harmonic we can estimate using the mean value property [34, I. Theorem 2.1] and (4.2.1)

$$\begin{aligned} |u_\lambda(x)| &= |u(\lambda x)| \\ &= \frac{1}{|B_{K(1-\lambda)|x|}(\lambda x)|} \left| \int_{B_{K(1-\lambda)|x|}(\lambda x)} u(t)dt \right| \\ &\le \frac{1}{|B_{K(1-\lambda)|x|}(\lambda x)|} \int_{B_{|x|((1-\lambda)+K(1-\lambda))}(x)} |\tilde{u}(t)|dt \\ &\le \frac{(K+1)^3}{K^3} \frac{1}{|B_{(K+1)(1-\lambda)|x|}(x)|} \int_{B_{|x|(1-\lambda)+K(1-\lambda))}(x)} |\tilde{u}(t)|dt \le cM\tilde{u}(x). \end{aligned}$$

Since M, the maximal operator in $L^q_w(\Omega)$ is bounded by Theorem 3.2.3, one has $M\tilde{u} \in L^q_w(\mathbb{R}^n)$. Thus, we have found a majorant. Moreover, since the harmonic function $u \in C^\infty(\Omega)$, the convergence $u_\lambda \to u$ is pointwise (and hence $|u_\lambda - u|^q w \to 0$ almost everywhere). By Lebesgue's Theorem we find $u_\lambda \to u$ in $L^q_w(\Omega)$. □

4.3 The Problem $\operatorname{div} u = f$

Throughout this section let $1 < q < \infty$ and $w \in A_q$.

Theorem 4.3.1. *Let $\Omega \subset \mathbb{R}^n$, $n \ge 2$, be a bounded and locally lipschitzian domain. Assume $f \in W^{k,q}_{w,0}(\Omega)$ such that $\int f = 0$. Then there exists a function $u \in W^{k+1,q}_{w,0}(\Omega)$ such that*

$$\operatorname{div} u = f \quad \textit{and} \quad \|u\|_{k+1,q,w} \le c\|f\|_{k,q,w},$$

where $c = c(\Omega, q, w, k)$. Moreover, u can be chosen such that it depends linearly on f and such that $u \in C^\infty_0(\Omega)$ if $f \in C^\infty_0(\Omega)$.

The proof follows the same lines as the unweighted case [7], [29, chapter III.3]. It uses non-translation-invariant singular integral operators. Thus we apply the following theorem proved in [49, V.6.13] which ensures the continuity of a certain class of such operators.

Theorem 4.3.2. *Let T be a bounded operator from $L^2(\mathbb{R}^n)$ into itself that is associated to a kernel K in the sense that*

$$(Tf)(x) = \int_{\mathbb{R}^n} K(x,y)f(y)dy$$

for all compactly supported $f \in L^2(\mathbb{R}^n)$ and all x outside the support of f. Suppose that for some $\gamma > 0$ and some $A > 0$ the kernel K satisfies the inequalities

$$|K(x,y)| \leq A|x-y|^{-n} \tag{4.3.1}$$

and

$$|K(x,y) - K(x',y)| \leq A\frac{|x-x'|^\gamma}{|x-y|^{n+\gamma}} \qquad \text{if } |x-x'| \leq \frac{1}{2}|x-y| \tag{4.3.2}$$

as well as the symmetric version of the second inequality in which the roles of x and y are interchanged. Writing

$$(T_\varepsilon f)(x) = \int_{|x-y|>\varepsilon} K(x,y)f(y)dy \quad \text{and} \quad (T_* f)(x) = \sup_{\varepsilon>0} |(T_\varepsilon f)(x)|,$$

we have that

$$\int [(T_* f)(x)]^q w(x)dx \leq c \int [(Mf)(x)]^q w(x)dx, \tag{4.3.3}$$

where f is bounded and has compact support, $w \in A_q$, and $1 < q < \infty$.

Since the maximal operator $M : L^q_w(\mathbb{R}^n) \to L^q_w(\mathbb{R}^n)$ is bounded, the inequality (4.3.3) guarantees that the sublinear operator T_* can be extended to a continuous sublinear operator $T_* : L^q_w(\Omega) \to L^q_w(\Omega)$.

However, to make use of the above theorem we have to modify the singular integral operator which appears in the proof of Lemma 4.3.3 below outside the bounded set Ω such that it possesses the properties assumed in Theorem 4.3.2.

In the proof of the following Lemma the occurring integral operators have to be understood in the Cauchy principle value sense $\lim_{\varepsilon\to 0} T_\varepsilon f$.

Lemma 4.3.3. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be bounded and star-shaped with respect to every point of some closed ball $\overline{B}$ with $\overline{B} \subset \Omega$.*

Then for every $f \in W^{k,q}_{w,0}(\Omega)$ with $\int_\Omega f = 0$ there exists a $v \in W^{k+1,q}_{w,0}(\Omega)$ with

$$\operatorname{div} v = f \quad \text{and} \quad \|v\|_{k+1,q,w} \leq c\|f\|_{k,q,w},$$

where $c = c(\Omega, q, w, k) > 0$, v depends linearly on f and $f \in C_0^\infty(\Omega)$ implies $v \in C_0^\infty(\Omega)$.

Proof. Without loss of generality we may assume, using a coordinate transformation, that $B = B_1(0)$.

First we assume that $f \in C_0^\infty(\Omega)$.

We choose $a \in C_0^\infty(B_1(0))$ such that $\int a = 1$ and define

$$v(x) := \int_\Omega f(y)(x-y)\left(\int_1^\infty a\left(y + \xi(x-y)\right)\xi^{n-1}d\xi\right)dy. \tag{4.3.4}$$

In the proof of [29, Lemma III.3.1] it is shown that $v \in C_0^\infty(\Omega)$ and $\operatorname{div} v = f$.

It thus remains to prove the weighted estimates. To do this we use the following representation of $\partial_j v$ also shown in the proof of [29, Lemma III.3.1]:

$$\begin{aligned}\partial_j v_i(x) = &\int_\Omega K_{i,j}(x, x-y)f(y)dy + f(x)\int_\Omega \frac{(x_j - y_j)(x_i - y_i)}{|x-y|^2}a(y)dy \\ =: &\, F_1(x) + F_2(x),\end{aligned} \tag{4.3.5}$$

where

$$\begin{aligned}K_{i,j}(x, x-y) = &\frac{\delta_{i,j}}{|x-y|^n}\int_0^\infty a\left(x + r\frac{x-y}{|x-y|}\right)(|x-y| + r)^{n-1}dr \\ &+ \frac{x_i - y_i}{|x-y|^{n+1}}\int_0^\infty \partial_j a\left(x + r\frac{x-y}{|x-y|}\right)(|x-y| + r)^n dr,\end{aligned} \tag{4.3.6}$$

for every $x, y \in \mathbb{R}^n$. To show the continuity of the integral operator $f \mapsto F_1$ its kernel must be modified. Set

$$E := \left\{z \in \Omega \mid z = \lambda z_1 + (1-\lambda)z_2,\ z_1 \in \operatorname{supp} f,\ z_2 \in \overline{B_1(0)},\ \lambda \in [0,1]\right\}.$$

Since Ω is star-shaped with respect to $\overline{B_1(0)}$, E is a compact subset of Ω. For $x \notin E$ and $y \in \operatorname{supp} f$ we have

$$x + r\frac{x-y}{|x-y|} \notin \overline{B} \quad \text{for all } r > 0,$$

which means $K_{i,j}(x, x-y) = 0$. Thus, if we choose a cut-off function $\psi \in C_0^\infty(\mathbb{R}^n)$ with $\psi(x) = 1$ on Ω and $\operatorname{supp}\psi \subset B_R(0)$ for some $R > 0$, and set $\varphi(x,y) = \psi(x)\psi(y)$ we obtain

$$f(y)K_{i,j}(x, x-y) = f(y)\varphi(x,y)K_{i,j}(x, x-y) =: f(y)\tilde{K}_{i,j}(x, x-y),$$

for $x, y \in \mathbb{R}^n$, if f is assumed to be extended by 0 to $\mathbb{R}^n$. Moreover, for $x \in B_R(0)$ we have

$$r > R+1 \;\Rightarrow\; \left|x + r\frac{x-y}{|x-y|}\right| \geq r - |x| > 1 \;\Rightarrow\; a\left(x + r\frac{x-y}{|x-y|}\right) = 0.$$

Thus for $x \in \Omega$ one has

$$\begin{aligned}\int_{\mathbb{R}^n} f(y)K_{i,j}(x, x-y)dy = &\int_{\mathbb{R}^n} f(y)\tilde{K}_{i,j}(x, x-y)dy \\ = &\int_{\mathbb{R}^n} f(y)\varphi(x,y)\left[\frac{\delta_{i,j}}{|x-y|^n}\int_0^{R+1} a\left(x + r\frac{x-y}{|x-y|}\right)(|x-y| + r)^{n-1}dr\right. \\ &\left. + \frac{x_i - y_i}{|x-y|^{n+1}}\int_0^{R+1}\partial_j a\left(x + r\frac{x-y}{|x-y|}\right)(|x-y| + r)^n dr\right]dy.\end{aligned}$$

Now we have to prove that $\tilde{K}_{i,j}$ satisfies the assumptions of Theorem 4.3.2. By the Calderón-Zygmund Theorem [8] we find that

$$f \mapsto \int_{\mathbb{R}^n} \psi(x) K_{i,j}(x, x-y) f(y) dy : \quad L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$$

is continuous. Since the multiplication M_ψ with the C_0^∞-function ψ is a continuous operator on $L^2(\mathbb{R}^n)$ we obtain the continuity of

$$\begin{aligned} f \mapsto &\int_{\mathbb{R}^n} \tilde{K}_{i,j}(x, x-y) f(y) dy \\ = &\int_{\mathbb{R}^n} \psi(x) K_{i,j}(x, x-y) M_\psi f(y) dy : \ L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n). \end{aligned}$$

It remains to prove the estimates (4.3.1) and (4.3.2). For (4.3.1) we may assume $|x|, |y| < R$. One has

$$\begin{aligned} &|x-y|^n |\tilde{K}_{i,j}(x, x-y)| \\ &= \left| \varphi(x,y) \delta_{i,j} \int_0^{R+1} a\left(x + r\frac{x-y}{|x-y|}\right) (|x-y|+r)^{n-1} dr \right. \\ &\quad \left. + \varphi(x,y) \frac{x_i - y_i}{|x-y|} \int_0^{R+1} \partial_j a\left(x + r\frac{x-y}{|x-y|}\right) (|x-y|+r)^n dr \right| \\ &\le c\left(\int_0^{R+1} (2R+r)^{n-1} dr + \int_0^{R+1} (2R+r)^n dr \right) = c'. \end{aligned}$$

To prove (4.3.2) we take $x, x', y \in \mathbb{R}^n$ with $|x - x'| \le \frac{1}{2}|x-y|$. If $(x,y), (x',y) \notin \operatorname{supp}\varphi$ nothing is to prove. Thus, without loss of generality we may assume that $y \le R$ and $x \le 3R$, since if $y \le R$ and $x \ge 3R$ then

$$|x'| \ge |x| - |x-x'| \ge |x| - \frac{1}{2}(|x|+|y|) \ge \frac{3}{2}R - \frac{1}{2}R = R.$$

Then using the triangle inequality together with the fact that a, φ and $(|x-y|+r)^n$ are Lipschitz continuous on compact sets we can estimate

$$\begin{aligned} &\left| \frac{x_i - y_i}{|x-y|^{n+1}} \varphi(x,y) \int_0^{R+1} \partial_j a\left(x + r\frac{x-y}{|x-y|}\right) (|x-y|+r)^n dr \right. \\ &\left. - \frac{x'_i - y_i}{|x'-y|^{n+1}} \varphi(x',y) \int_0^{R+1} \partial_j a\left(x' + r\frac{x'-y}{|x'-y|}\right) (|x'-y|+r)^n dr \right| \\ &\le \left| \left(\frac{x_i - y_i}{|x-y|^{n+1}} - \frac{x'_i - y_i}{|x'-y|^{n+1}} \right) \varphi(x,y) \int_0^{R+1} \partial_j a\left(x + r\frac{x-y}{|x-y|}\right) (|x-y|+r)^n dr \right| \\ &+ \left| \frac{x'_i - y_i}{|x'-y|^{n+1}} (\varphi(x,y) - \varphi(x',y)) \int_0^{R+1} \partial_j a\left(x + r\frac{x-y}{|x-y|}\right) (|x-y|+r)^n dr \right| \\ &+ \frac{|x'_i - y_i|}{|x'-y|^{n+1}} \varphi(x',y) \int_0^{R+1} \left| \partial_j a\left(x + r\frac{x-y}{|x-y|}\right) - \partial_j a\left(x' + r\frac{x'-y}{|x'-y|}\right) \right| \\ &\qquad\qquad (|x-y|+r)^n dr \\ &+ \frac{|x'_i - y_i|}{|x'-y|^{n+1}} \varphi(x',y) \int_0^{R+1} \left| \partial_j a\left(x' + r\frac{x'-y}{|x'-y|}\right) \right| |(|x-y|+r)^n - (|x'-y|+r)^n| dr \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned}$$

Using the Lipschitz continuity of $\partial_i a$ and $|x-y| \leq 4R$ we obtain

$$I_3 \leq \frac{c}{|x-y|^n} \int_0^{R+1} L\left(|x-x'| + \frac{|x-x'|}{|x-y|}\right)(|x-y|+r)^n dr \leq c\frac{|x-x'|}{|x-y|^{n+1}}.$$

I_2 and I_4 can be estimated analogously. For I_1 we estimate

$$\begin{aligned}
\left|\frac{x_i-y_i}{|x-y|^{n+1}} - \frac{x_i'-y_i}{|x'-y|^{n+1}}\right| &\leq \frac{|x_i-x_i'|}{|x-y|^{n+1}} + \left|\frac{1}{|x-y|^{n+1}} - \frac{1}{|x'-y|^{n+1}}\right| |x_i'-y_i| \\
&\leq \frac{|x-x'|}{|x-y|^{n+1}} + \left|\frac{|x'-y|^{n+1} - |x-y|^{n+1}}{|x-y|^{n+1}|x'-y|^{n+1}}\right| |x_i'-y_i| \\
&\leq \frac{|x-x'|}{|x-y|^{n+1}} + c\frac{\big||x'-y|-|x-y|\big| \cdot |x-y|^n}{|x-y|^{2n+2}} |x_i'-y_i| \\
&\leq c\frac{|x-x'|}{|x-y|^{n+1}},
\end{aligned}$$

where we used that $|x'-y| \geq \frac{1}{2}|x-y|$. The estimate $\big||x'-y|^{n+1} - |x-y|^{n+1}\big| \leq c\big||x'-y|-|x-y|\big| \cdot |x-y|^n$ follows from an elementary induction with respect to n.

The first summand in (4.3.6) can treated in the same way. Moreover, interchanging the roles of x and y the same kind of estimates can be done.

Combining the above and using Theorem 4.3.2 we obtain

$$\|F_1\|_{q,w} \leq \|T^* f\|_{q,w} \leq c\|Mf\|_{q,w} \leq c\|f\|_{q,w}$$

where T^* is the operator given by Theorem 4.3.2 and associated to the kernel $\tilde{K}_{i,j}$. The function F_2 appearing in (4.3.5) is easily estimated since

$$\int_\Omega \frac{(x_j-y_j)(x_i-y_i)}{|x-y|^2} a(y)dy$$

is bounded. Thus using the Poincaré inequality (3.3.1) we obtain $\|v\|_{1,q,w} \leq c\|f\|_{q,w}$.

Now the general case with $f \in L^q_w(\Omega)$ follows easily, since we can approximate f by C_0^∞-functions (f_n) with $\int f_n = 0$.

It remains to prove the estimate in the spaces $W^{k,q}_w(\Omega)$. Using Leibniz' formula one can show (see [29, Remark III.3.2])

$$\partial^\alpha v(x) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \int_\Omega N_\beta(x,y) \partial^{\alpha-\beta} f(y)dy,$$

where

$$N_\beta(x,y) = (x-y)\int_1^\infty \partial^\beta a(y + r(x-y)) r^{n-1} dr.$$

Clearly $\partial^\beta a \in C_0^\infty(B_1(0))$. Hence the same proof as above yields

$$\|\partial^\alpha v\|_{1,q,w} \leq c\|f\|_{k,q,w}$$

for $f \in C_0^\infty(\Omega)$ and every α with $|\alpha| \leq k$. Approximating an arbitrary $f \in W^{k,q}_{w,0}(\Omega)$ with $\int f = 0$ by C_0^∞-functions (f_n) with $\int f_n = 0$ finishes the proof. $\square$

The following Lemma is the weighted analogue to [29, Lemma III.3.4.]. Its proof works in exactly the same way as in the case of unweighted function spaces.

Lemma 4.3.4. *Let Ω be a bounded and locally lipschitzian domain.*

1. *There exist open sets $\Omega_1, ..., \Omega_m$ with $\Omega = \bigcup_{i=1}^m \Omega_i$ such that each Ω_i is star-shaped with respect to an open ball B_i with $\overline{B}_i \subset \Omega_i$.*

2. *For every $f \in C_0^\infty(\Omega)$ with $\int_\Omega = 0$ there exist $f_i \in C_0^\infty(\Omega_i)$, $i = 1, ..., m$, with $f = \sum_{i=1}^m f_i$, $\int f_i = 0$ and $\|f_i\|_{k,q,w} \leq c\|f\|_{k,q,w}$ for every $k \in \mathbb{N}_0$ and $q \geq 1$, $c = c(k, q, w, \Omega)$.*

We now prove Theorem 4.3.1.

Proof. Let $f \in C_0^\infty(\Omega)$ with $\int f = 0$ and take Ω_i, f_i, $i = 1, ..., m$, as in Lemma 4.3.4. We denote by v_i the solution to $\operatorname{div} v_i = f_i$ given by Lemma 4.3.3. Then we have

$$\|v_i\|_{k+1,q,w} \leq c\|f_i\|_{k,q,w} \leq c\|f\|_{k,q,w}.$$

Then $v = \sum_{i=1}^m v_i$ solves $\operatorname{div} v = f$ with $\|v\|_{k+1,q,w} \leq c\|f\|_{k,q,w}$. For arbitrary $f \in W_{w,0}^{k,q}(\Omega)$ with $\int f = 0$ use again approximations with C_0^∞-functions. □

4.4 The Helmholtz Decomposition

For a bounded $C^{1,1}$-domain Ω, $q \in (1, \infty)$ and $w \in A_q$ we set

$$\begin{aligned} L_{w,\sigma}^q(\Omega) &:= \{u \in L_w^q(\Omega) \mid \langle u, \nabla\phi\rangle = 0 \text{ for all } \phi \in W_{w'}^{1,q'}(\Omega)\} \quad \text{and} \\ G_w^q(\Omega) &:= \{\nabla p \mid p \in W_w^{1,q}(\Omega)\}. \end{aligned} \tag{4.4.1}$$

Then by [24] the following assertions hold true.

- $L_{w,\sigma}^q(\Omega) = \overline{C_{0,\sigma}^\infty(\Omega)}^{L_w^q(\Omega)}$. This decomposition of $L_w^q(\Omega)$ is called Helmholtz decomposition.

- $L_w^q(\Omega) = L_{w,\sigma}^q(\Omega) \oplus G_w^q(\Omega)$.

- There exists a continuous projection

$$P_{q,w} : L_w^q(\Omega) \to L_{w,\sigma}^q(\Omega)$$

  with image space $L_{w,\sigma}^q(\Omega)$ and kernel $G_w^q(\Omega)$, the so-called Helmholtz projection.

- If $u = P_{q,w}u + \nabla p$, then p is the solution to the weak Neumann problem

$$\langle \nabla p, \nabla\phi\rangle_\Omega = \langle u, \nabla\phi\rangle_\Omega \quad \text{for every} \quad \phi \in W_{w'}^{1,q'}(\Omega).$$

In Section 9.5 we need some higher regularity of $P_{q,w}f$ in the case of higher regularity of f. However, this requires stronger assumptions on the smoothness of the boundary of Ω.

Theorem 4.4.1. *Let $k = 1, 2$ and $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded $C^{k,1}$-domain. Then the Helmholtz projection*

$$P_{q,w} : W_w^{k,q}(\Omega) \to W_w^{k,q}(\Omega)$$

is continuous.

Proof. See Appendix A.2 □

5 Extension Theorems

5.1 Appropriate Charts

A domain Ω is called a $C^{k,1}$-domain, if the boundary can locally be expressed as the graph of a $C^{k,1}$-function, i.e, for every $x_0 \in \partial\Omega$ we can rotate and shift the coordinate system such that in a neighborhood $U(x_0)$ of x_0 one has

$$\partial\Omega \cap U(x_0) = \{(x', a(x')) \mid x' \in V(0)\}, \tag{5.1.1}$$

where $V(0)$ is an appropriate ($(n-1)$-dimensional) neighborhood of 0 and $a : V(0) \to \mathbb{R}$ is a $C^{k,1}$-function.

In this case the function a and the coordinate system can be chosen such that $(0, a(0)) = x_0$ and $\nabla a(0) = 0$.

For the definition of the boundary values of very weak solutions we need appropriate extension theorems. The proof of them requires a chart α for which one has

$$\frac{\partial}{\partial x_n}\alpha(x', 0) = -N(x'), \tag{5.1.2}$$

i.e., normals to the boundary of the half space are mapped to normals to $\partial\Omega$. The natural mapping with this property would be

$$x = (x', x_n) \mapsto \begin{pmatrix} x' \\ a(x') \end{pmatrix} - x_n \cdot N(x'),$$

where $N(x)$ stands for the outer unit normal vector at $(x', a(x'))$. Such charts are used by Nečas [39]. However, if a is a $C^{k,1}$-function, then, since it includes the outer normal N, this chart is only of class $C^{k-1,1}$. For this reason we introduce a different chart which conserves the regularity and still has the property (5.1.2).

Lemma 5.1.1. *For $k \in \mathbb{N}$ let $\Omega \subset \mathbb{R}^n$ be a $C^{k,1}$-domain. Then for every $x_0 \in \partial\Omega$ there exists a neighborhood U of x_0, a neighborhood V of 0 and a bijective map $\alpha : V \to U$ such that*

$$\alpha(0) = x_0, \quad \alpha(V \cap (\mathbb{R}^{n-1} \times \{0\})) = U \cap \partial\Omega, \quad \alpha(V \cap \mathbb{R}^n_+) = U \cap \Omega$$

and with the following properties:

1. $\alpha \in C^{k,1}(V, U)$,
2. $\frac{\partial}{\partial x_n}\alpha(x', 0) = -N(x')$ *and* $\left(\frac{\partial}{\partial x_n}\right)^j \alpha(x', 0) = 0$ *for* $j \geq 2$ *even.*
3. *With the notation of (5.1.1) one has*

a) $\|\alpha\|_{C^{k,1}(V,U)}$ *can be estimated by* $\|a\|_{C^{k,1}(V\cap(\mathbb{R}^{n-1}\times\{0\}))}$.

b) *There exists some* $r > 0$ *which only depends on the sets* $U(x_0)$, $V(x_0)$ *and the size of* $\|a\|_{C^{k,1}(V\cap(\mathbb{R}^{n-1}\times\{0\}))}$ *such that* $B_r(x_0) \subset U$.

Proof. We use the notation $x = (x', x_n)$ with $x' \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$ and $\partial^\gamma = \partial_{x_1}^{\gamma_1} \cdot \ldots \cdot \partial_{x_n}^{\gamma_1}$ for $\gamma \in \mathbb{N}^n$.

Let $0 \leq \rho \in C_0^\infty(\mathbb{R}^{n-1})$ be radially symmetric such that $\operatorname{supp}\rho \subset B_1(0)$ and $\int_{\mathbb{R}^{n-1}} \rho = 1$. For $t \neq 0$ we set $\rho_t(x') = \frac{1}{|t|^{n-1}}\rho(\frac{x'}{t})$. We define the function α as follows:

$$\alpha(x', x_n) = \begin{cases} \binom{x'}{a(x')} - (x_n\rho_{x_n} * N)(x') & \text{if } \ x_n \neq 0 \\ \binom{x'}{a(x')} & \text{if } \ x_n = 0, \end{cases}$$

where the convolution takes place in $\mathbb{R}^{n-1}$. Then for every multi index $\gamma = (\gamma', \gamma_n) \in \mathbb{N}_0$, with $|\gamma| \leq k$ and $|\gamma'| < k$ one has for $x_n \neq 0$

$$\begin{aligned}
\partial^\gamma(x_n\rho_{x_n} * N)(x') =& \partial^{(0,\gamma_n)}(x_n\rho_{x_n} * \partial^{\gamma'} N(x')) \\
=& \partial^{(0,\gamma_n)} x_n \int_{\mathbb{R}^{n-1}} \rho(\xi)\partial^{\gamma'} N(x' - x_n\xi) d\xi \\
=& \gamma_n(-1)^{\gamma_n - 1} \int \rho(\xi)\nabla^{\gamma_n-1}\partial^{\gamma'} N(x' - x_n\xi) \underbrace{(\xi, ..., \xi)}_{\gamma_n - 1} d\xi \\
& + x_n \frac{\partial}{\partial x_n}\left((-1)^{\gamma_n-1} \int \rho(\xi)\nabla^{\gamma_n-1}\partial^{\gamma'} N(x' - x_n\xi) \underbrace{(\xi, ..., \xi)}_{\gamma_n - 1} d\xi \right).
\end{aligned}$$

Then using change of variables an the fact that the map $\nabla^{\gamma_n-1}\partial^{\gamma'} N(x' - \xi)$ is $(\gamma_n - 1)$-linear, the second summand is equal to

$$\begin{aligned}
& (-1)^{\gamma_n-1} x_n \frac{\partial}{\partial x_n} \int \frac{1}{|x_n|^{n+\gamma_n-2}} \rho\left(\frac{\xi}{x_n}\right) \nabla^{\gamma_n-1}\partial^{\gamma'} N(x' - \xi)(\xi, ..., \xi) d\xi \\
&= (-1)^{\gamma_n-1} \int \left(\frac{-n - \gamma_n + 2}{|x_n|^{n+\gamma_n-2}} \rho\left(\frac{\xi}{x_n}\right) + \frac{-1}{|x_n|^{n+\gamma_n-1}} \nabla\rho\left(\frac{\xi}{x_n}\right) \cdot \xi \right) \\
& \qquad \cdot \nabla^{\gamma_n-1}\partial^{\gamma'} N(x' - \xi)(\xi, ..., \xi) d\xi \\
&= (-1)^{\gamma_n-1} \int ((-n - \gamma_n + 2)\rho(\xi) - \nabla\rho(\xi) \cdot \xi) \, \nabla^{\gamma_n-1}\partial^{\gamma'} N(x' - \xi x_n) \underbrace{(\xi, ..., \xi)}_{\gamma_n - 1} d\xi.
\end{aligned}$$

Hence

$$\begin{aligned}
\partial^\gamma(x_n\rho_{x_n} * N)(x') =& (-1)^{\gamma_n-1} \int ((-n + 2)\rho(\xi) - \nabla\rho(\xi) \cdot \xi) \\
& \cdot \nabla^{\gamma_n-1}\partial^{\gamma'} N(x' - \xi x_n) \underbrace{(\xi, ..., \xi)}_{\gamma_n - 1} d\xi.
\end{aligned} \tag{5.1.3}$$

Still we have to consider the case $|\gamma'| = k$ in which the situation is easier. Integration by parts yields

$$\begin{aligned}\partial^\gamma(x_n\rho_{x_n} * N)(x') &= x_n\partial^{\beta_2}\int \rho(\xi)\partial^{\beta_1}N(x'-x_n\xi)d\xi \\ &= x_n\partial^{\beta_2}\int \rho(\xi)\frac{-1}{x_n}\partial_\xi^{\beta_1}(N(x'-x_n\xi))d\xi \\ &= \int \partial^{\beta_1}\rho(\xi)\partial^{\beta_2}N(x'-x_n\xi)d\xi,\end{aligned} \tag{5.1.4}$$

where $\gamma = \beta_1 + \beta_2$ and $|\beta_1| = 1$.

The map $x \mapsto \binom{x'}{a(x')}$ is of type $C^{k,1}$ because a is. It remains to show that $\partial^\gamma(x_n\rho_{x_n} * N(x'))$ is Lipschitz continuous for every $\gamma \in \mathbb{N}^n$, $|\gamma| \le k$. This is an easy consequence of the representations (5.1.3) and (5.1.4) and of $N \in C^{k-1,1}$, e.g.,

$$\begin{aligned}&|\partial^\gamma(x_n\rho_{x_n} * N(x')) - \partial^\gamma(y_n\rho_{y_n} * N(y'))| \\ &\le \int_{B_1(0)} |c_n\rho(\xi) - \nabla\rho(\xi)\cdot\xi|\,|\nabla^{\gamma_n-1}\partial^{\gamma'}N(x'-\xi x_n) - \nabla^{\gamma_n-1}\partial^{\gamma'}N(y'-\xi y_n)|d\xi \\ &\le cL_\gamma \sup_{\xi\in B_1(0)} |x'-y'+\xi(x_n-y_n)| \le cL_\gamma|x-y|.\end{aligned}$$

A similar calculation shows that the right hand side of (5.1.4) is Lipschitz continuous.

It remains to show that

$$\begin{aligned}\tfrac{\partial}{\partial x_n}\alpha(x',0) &= -N(x') && \text{and} \\ \left(\tfrac{\partial}{\partial x_n}\right)^j\alpha(x',0) &= 0, && \text{for } \ j \ge 2 \ \text{ even.}\end{aligned}$$

From (5.1.3) we have for $j > 1$ even

$$\begin{aligned}\left(\frac{\partial}{\partial x_n}\right)^j\alpha(x',0) &= \left(\frac{\partial}{\partial x_n}\right)^j\begin{pmatrix}x'\\a(x')\end{pmatrix} - \left(\frac{\partial}{\partial x_n}\right)^j(x_n\rho_{x_n} * N)(x') \\ =&(-1)^{j-1}\lim_{x_n\to 0}\int((-n+2)\rho(\xi) - \nabla\rho(\xi)\cdot\xi)\,\nabla^{j-1}N(x'-\xi x_n)\underbrace{(\xi,\dots,\xi)}_{j-1}d\xi \\ =&(-1)^{j-1}\int((-n+2)\rho(\xi) - \nabla\rho(\xi)\cdot\xi)\nabla^{j-1}N(x')(\xi,\dots,\xi)d\xi \\ =&(-1)^{j-1}\Bigg(\int(-n+2)\rho(\xi)\nabla^{j-1}N(x')(\xi,\dots,\xi)d\xi \\ &+\sum_{j=1}^{n-1}\int\rho(\xi)\left[\nabla^{j-1}N(x')(\xi,\dots,\xi) + (j-1)\xi_k\nabla^{j-1}N(x')(e_k,\xi,\dots,\xi)\right]d\xi\Bigg) \\ =&(-1)^{j-1}j\int\rho(\xi)\nabla^{j-1}N(x')(\xi,\dots,\xi)d\xi = 0\end{aligned}$$

since ρ is presumed to be rotationally symmetric and $\xi \mapsto \nabla^{j-1}N(x')(\xi,\dots,\xi)$ is an odd function for $j-1$ odd. Similarly,

$$\begin{aligned}\frac{\partial}{\partial x_n}\alpha(x',0) &= -N(x')\left((2-n)\int \rho(\xi)d\xi - \sum_{i=1}^{n-1}\int \partial_i\rho(\xi)\xi_i d\xi\right)\\ &= -\left((2-n)N(x') + \sum_{i=1}^{n-1} N(x')\right) = -N(x').\end{aligned}$$

It remains to show 3. b).

By (5.1.3) and (5.1.4) one has, since $\nabla a(0)=0$ and $N(0)=-e_n$,

$$\begin{aligned}\nabla\alpha(0) &= \nabla\begin{pmatrix} x' \\ a(x')\end{pmatrix} - \nabla(x_n\rho_{x_n} * N(x'))\big|_{x=0}\\ &= \begin{pmatrix} 1 & 0 & \cdots & 0\\ 0 & \ddots & \ddots & 0\\ 0 & 0 & 1 & \vdots\\ & \nabla a(0) & & 0\end{pmatrix} + \begin{pmatrix} & & & \\ & 0 & & \\ & & & \\ & & & 1\end{pmatrix} = \mathrm{id}\,.\end{aligned}$$

Since $\nabla\alpha$ is Lipschitz continuous with constant K, we get for $x,y\in\overline{B_r(0)}$, $r<\frac{1}{2K}$, that

$$\begin{aligned}|\alpha(x)-\alpha(y)| &= \sup_{|v|=1}|v\cdot\nabla\alpha(\xi_v)(x-y)|, \qquad \xi_v\in\{(1-t)x+ty\,|\,t\in(0,1)\}\\ &\geq \inf_{\xi\in B_r(0)}\left|\frac{x-y}{|x-y|}\nabla\alpha(\xi)(x-y)\right|\\ &\geq \inf_{\xi\in B_r(0)}\left(\frac{|x-y|^2}{|x-y|} - \left|(x-y)(\nabla\alpha(\xi)-\nabla\alpha(0))\frac{x-y}{|x-y|}\right|\right)\\ &> \frac{1}{2}|x-y|.\end{aligned}$$

From this inequality it immediately follows that α is injective on $B_r(0)$.

Moreover, it is easily seen that $B_{\frac{r}{2}}(x_0)\subset\alpha B_r(0)$. Indeed for $x\in\partial B_r(0)$ one has $|\alpha(x)-x_0|>\frac{1}{2}|x-0|$. Since $\nabla\alpha(x)$ is invertible for every $x\in B_r(0)$ it follows from the Inverse Function Theorem that $\alpha(B_r(0))$ is open. Together with the continuity of α we obtain

$$B_{\frac{r}{2}}(x_0)\cap\partial\alpha(B_r(0)) = B_{\frac{r}{2}}(x_0)\cap\alpha(\partial B_r(0)) = \emptyset.$$

Assume now that $y\in B_{\frac{r}{2}}(x_0)\setminus\alpha(B_r(0))$. Then the straight line from y to x_0 intersects $\partial\alpha(B_r(0))$. Thus this intersection point is contained in the intersection which we have shown to be empty. This is a contradiction.

This argument finishes the proof. □

5.2 Extension of Normal Derivatives

Our next objective is to construct a linear extension operator that maps functions defined on the boundary $\partial\Omega$ to a function defined on the domain Ω whose boundary values or normal derivatives are the given preimages.

We start with the half space.

Theorem 5.2.1. *Let* $1 < q < \infty$, $w \in A_q$ *and* $k \in \mathbb{N}$. *Then there exists a continuous linear operator*

$$T : \prod_{j=0}^{k-1} T_w^{k-j,q}(\mathbb{R}^{n-1}) \to W_w^{k,q}(\mathbb{R}_+^n)$$

such that $(-1)^j \frac{\partial^j}{\partial x_n^j} T(g_0, ..., g_{k-1})|_{x_n=0} = g_j$, $j = 0, ..., k-1$.

Proof. It suffices to show that for every $g \in T_w^{k-j,q}(\mathbb{R}^{n-1})$, $j = 0, ..., k-1$, there exists a $u \in W_w^{k,q}(\mathbb{R}_+^n)$ depending continuously and linearly on g such that $\frac{\partial^j}{\partial x_n^j} u = g$ and $\frac{\partial^i}{\partial x_n^i} u = 0$ for every $i = 0, ..., j-1$. To see this assume that for every $j = 0, ..., k-1$ there exists a continuous linear operator

$$T_j : T_w^{k-j,q}(\mathbb{R}^{n-1}) \to W_w^{k,q}(\mathbb{R}_+^n), \quad \frac{\partial^i}{\partial x_n^i} T_j(h)\big|_{x_n=0} = \begin{cases} 0, & \text{if } i < j \\ h, & \text{if } i = j \end{cases}.$$

For $g = (g_0, ..., g_{k-1})$ we can define $S_0(g) := T_0(g)$ and

$$S_{j+1}(g) := S_j(g) + T_{j+1}\left(g_{j+1} - \frac{\partial^{j+1}}{\partial x_n^{j+1}} S_j(g)\right).$$

Then $T = S_{k-1}$ solves our problem.

Next we show the weaker assertion. For $g \in T_w^{k-j,q}(\mathbb{R}^{n-1})$ let $v \in W_w^{k-j,q}(\mathbb{R}_+^n)$ with $(1-\Delta)v = 0$ and $v|_{\mathbb{R}^{n-1}} = g$ which is uniquely defined by [27, Theorem 4.4.] and Theorem 4.1.2. Let $\zeta \in C^\infty(\mathbb{R}_+)$ be a cut-off function with $\zeta(t) = 1$ for $t < 1$ and $\zeta(t) = 0$ for $t > 2$. We set

$$\phi(x) = \phi(x_n) = \frac{1}{j!} x_n^j \cdot \zeta(x_n) \quad \text{and} \quad u(x) = \phi(x)v(x). \tag{5.2.1}$$

We show that ϕu solves the problem. More precisely we prove the following claim:

If $\phi \in C^\infty(\overline{\mathbb{R}_+^n})$ with $\phi(x) = \phi(x_n)$, $\operatorname{supp} \phi \subset \mathbb{R}^{n-1} \times [0,2]$ and $(\frac{\partial}{\partial x_n})^m \phi|_{x_n=0} = 0$ for $m = 0, ..., l$ and $v \in W_w^{k,q}(\mathbb{R}_+^n)$ with $(1-\Delta)v = 0$ then $\phi v \in W_w^{k+l,q}(\mathbb{R}_+^n)$ with $\|\phi v\|_{k+l,q,w} \le c\|v\|_{k,q,w}$.

To prove this we use mathematical induction with respect to l and assume that we already know the assertion is true for $l-1$, $l-2$ and all k.

Since $(1-\Delta)v = 0$ we obtain

$$(1-\Delta)(\phi v) = -\Delta\phi v - 2\nabla v \cdot \nabla\phi. \tag{5.2.2}$$

As $(\frac{\partial}{\partial x_n})^m \Delta\phi|_{x_n=0} = 0$ for $m = 0, ..., l-2$, $(\frac{\partial}{\partial x_n})^m \nabla\phi|_{x_n=0} = 0$ for $m = 0, ..., l-1$ and $(1-\Delta)\nabla v = 0$, (5.2.2) and the induction hypothesis yield $(1-\Delta)(\phi v) \in W_w^{k+l-2,q}(\mathbb{R}_+^n)$. Thus and since $\phi v|_{\mathbb{R}^{n-1}} = 0$, one has $\phi v \in W_w^{k+l,q}(\mathbb{R}_+^n)$ by the regularity of the Laplace resolvent problem. Moreover

$$\|\phi v\|_{k+l,q,w} \le c\|(\Delta\phi)v + 2\nabla v \nabla\phi\|_{k+l-2,q,w} \le c(\|v\|_{k,q,w} + \|\nabla v\|_{k-1,q,w}) \le c\|v\|_{k,q,w}.$$

For the start of induction we need the cases $l = 0$ and $l = 1$. The case $l = 0$ is trivial, the case $l = 1$ is proved in the same way as the induction step.

If one applies the above claim to $u\phi v$ given by (5.2.1) we get $u \in W_w^{k,q}(\Omega)$. Moreover,

$$\frac{\partial^l}{\partial x_n^l} u(x', 0) = \sum_{\nu=0}^{l} \binom{l}{\nu} \frac{\partial^\nu}{\partial x_n^\nu} v \frac{\partial^{l-\nu}}{\partial x_n^{l-\nu}} \phi(x', 0) = \begin{cases} 0 & \text{if } l < j \\ g(x') & \text{if } l = j. \end{cases}$$

This shows the assertion about the boundary values. □

Theorem 5.2.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded $C^{k-1,1}$-domain, $k \geq 1$. Then there exists a continuous linear operator*

$$L : \prod_{j=0}^{k-1} T_w^{k-j,q}(\partial\Omega) \to W_w^{k,q}(\Omega)$$

such that $\frac{\partial^j}{\partial N^j} L(g)|_{\partial\Omega} = g_j$, $0 \leq j \leq k-1$, where $g = (g_0, ..., g_{k-1})$.

Proof. As in the proof of Theorem 5.2.1 we construct an operator

$$L_j : T_w^{k-j,q}(\partial\Omega) \to W_w^{k,q}(\Omega), \quad \frac{\partial^k}{\partial N^j} L_j(g) = \begin{cases} g & \text{if } k = j \\ 0 & \text{if } k < j. \end{cases}$$

Then the general case follows as in the proof of Theorem 5.2.1.

We choose the collection of charts $(\alpha_i, V_i, U_i)_{i=1}^m$ according to Lemma 5.1.1 and a decomposition of unity $(\phi_i)_{i=1}^m$ subordinate to the covering $\{U_i\}$.

To simplify the notation we fix i and set $\gamma = \alpha_i$, $U = U_i$, $V = V_i$ and $\phi = \phi_i$. Moreover, for $g \in T_w^{k-j,q}(\partial\Omega)$ we set $\tilde{g} = (g \cdot \phi) \circ \gamma$. By Lemma 3.3.6 we know $\tilde{g}_j \in T_{w\circ\gamma}^{k-j,q}(\mathbb{R}^{n-1})$. Thus we may apply the operator T from Theorem 5.2.1 and set

$$v := v_i := L_{i,j}(g) := (\psi_i T(0, ..., 0, \tilde{g}, 0, ..., 0)) \circ \gamma^{-1},$$

meaning, that the j'th component of $(0, ..., 0, \tilde{g}, 0, ..., 0)$ is $\tilde{g}$.

Moreover, $(\psi_i)_i \subset C_0^\infty(\overline{\mathbb{R}}_+^n)$ with $\psi_i = 1$ in a neighborhood of $\operatorname{supp} \tilde{g}$ and $\operatorname{supp} \psi_i \subset V_i$. Here ψ_i can be chosen such that $\frac{\partial^k}{\partial x_n^k} \psi_i(x', 0) = 0$ for every $k \in \mathbb{N}$.

Then we have by the choice of γ according to Lemma 5.1.1 for every $k \leq j$.

$$\begin{aligned}
(-1)^k \delta_{j,k} \tilde{g}(x') &= \frac{\partial^k}{\partial x_n^k} T(...0, \tilde{g}, 0...)(x', 0) \\
&= \frac{\partial^k}{\partial x_n^k} \psi_i T(...0, \tilde{g}, 0...)(x', 0) \\
&= \left(\frac{\partial^k}{\partial x_n^k} (v \circ \gamma) \right) (x', 0) = \frac{\partial^{k-1}}{\partial x_n^{k-1}} (\nabla v \circ \gamma) \cdot \partial_n \gamma(x', 0) \\
&= (\nabla^k v \circ \gamma) \cdot (\partial_n \gamma, ..., \partial_n \gamma)(x', 0) + \text{terms containing } \nabla^i v \circ \gamma(x', 0),\ i < j \\
&= (\nabla^k v(\gamma(x', 0)))(\underbrace{-N(x'), ..., -N(x')}_{k}) = (-1)^k \left(\frac{\partial^k}{\partial^k N} v \right) (\gamma(x', 0)).
\end{aligned}$$

The terms containing $\nabla^i v \circ \gamma(x', 0)$ vanish for $i < j$, since

$$\nabla^i (v \circ \gamma)(x', 0) = \nabla^i (\psi_i T(0, ..., 0, \tilde{g}, 0, ..., 0))(x', 0) = 0$$

for $i = 1, ..., j-1$ by the definition of T.

Finally we set

$$L_j(g) = \sum_{i=1}^{m} L_{i,j}(g),$$

and obtain

$$\frac{\partial^k}{\partial N^k} L_j(g)|_{\partial\Omega} = \sum_{i=1}^{m} \frac{\partial^k}{\partial N^k} L_{i,j}(g)|_{\partial\Omega} = \begin{cases} g & \text{if } k = j \\ 0 & \text{if } k < j. \end{cases}$$

The continuity of L_j follows from Lemma 3.3.6 and the continuity of T in Theorem 5.2.1, more precisely,

$$\begin{aligned} \|L_j(g)\|_{k,q,w,\Omega}^q &= \left\| \sum_{i=1}^{m} (\psi_i T(...0, \tilde{g}, 0...)) \circ \alpha_i^{-1} \right\|_{k,q,w,\Omega}^q \\ &\le c \sum_{i=1}^{m} \|\phi_i g\|_{T_w^{k-j,q}(\partial\Omega)}^q \le c\|g\|_{T_w^{k-j,q}(\partial\Omega)}^q \end{aligned}$$

for $j = 0, ..., k-1$. □

Corollary 5.2.3. *Let Ω be a bounded $C^{1,1}$-domain. Then*

$$W_{w,0}^{2,q}(\Omega) = \left\{ u \in W_w^{2,q}(\Omega) \mid u|_{\partial\Omega} = 0, \ \nabla u|_{\partial\Omega} = 0 \right\}.$$

Proof. The inclusion "$\subset$" is clear by the continuity of the trace.

In the same way as in [14, 5.5. Theorem 2] one shows the assertion "$\supset$" for the unweighed case $w = 1$.

Let $u \in W_w^{2,q}(\Omega)$, $u|_{\partial\Omega} = 0$ and $\nabla u|_{\partial\Omega} = 0$. We have to approximate u by a sequence in $C_0^\infty(\Omega)$. To do this, take a sequence $(v_n) \subset C^\infty(\overline{\Omega})$ converging to u in $W_w^{2,q}(\Omega)$.

Then

$$v_n|_{\partial\Omega} \xrightarrow{T_w^{2,q}(\partial\Omega)} 0 \ \text{ and } \ \nabla v_n|_{\partial\Omega} \xrightarrow{T_w^{1,q}(\partial\Omega)} 0.$$

Let

$$L : T_w^{1,q}(\partial\Omega) \times T_w^{2,q}(\partial\Omega) \to W_w^{2,q}(\Omega)$$

be the operator from Theorem 5.2.2 and set

$$u_n := v_n - L(v_n|_{\partial\Omega}, N \cdot \nabla v_n|_{\partial\Omega}).$$

By construction, the operator L is independent of q and w. Thus, since every v_n is smooth, we obtain

$$u_n \in W^{2,r}(\Omega), \quad u_n|_{\partial\Omega} = 0, \quad \text{and} \ \ \nabla u_n|_{\partial\Omega} = 0 \ \text{ and } \ u_n \xrightarrow{W_w^{2,q}(\Omega)} u,$$

where $r > 1$ is chosen such that $L^r(\Omega) \hookrightarrow L_w^q(\Omega)$. Then, by the unweighted case, there exists a sequence $(\phi_k^{(n)})_k \subset C_0^\infty(\Omega)$ converging to u_n in $W^{2,r}(\Omega)$. This guarantees the existence of a sequence in $C_0^\infty(\Omega)$ converging to u in $W_w^{2,q}(\Omega)$. □

6 The Stokes Problem with Irregular Data

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 2$, with $C^{1,1}$-boundary and let $1 < q < \infty$ and $w \in A_q$. The aim of this chapter is to find a class of solutions to the Stokes resolvent problem in weighted Lebesgue and Sobolev spaces demanding as low regularity of the data as possible.

In Section 6.1 the divergence and external force are so irregular that it is impossible to speak of boundary values. In Section 6.2 we show higher regularity of the solution in the case of higher regularity of the data. Moreover, it will be shown that this class of solutions includes strong solutions. In Section 6.3 it is presented how one can explain boundary values in the case that the data is regular enough such that it can be decomposed into distributions on Ω and on $\partial\Omega$.

6.1 Very Weak Solutions Concerning Non-Distributional Data

We consider the stationary Stokes resolvent problem with inhomogeneous data

$$\begin{aligned} \lambda u - \Delta u + \nabla p &= F && \text{in } \Omega \\ \operatorname{div} u &= K && \text{in } \Omega \\ u &= g && \text{on } \partial\Omega. \end{aligned} \tag{6.1.1}$$

If one multiplies the first equations in (6.1.1) with a solenoidal test function ϕ vanishing on the boundary, then formal integration by parts yields

$$\langle u, \lambda\phi \rangle - \langle u, \Delta\phi \rangle_\Omega = \langle F, \phi \rangle_\Omega - \langle g, N \cdot \nabla\phi \rangle_{\partial\Omega}. \tag{6.1.2}$$

Applying the same method to the second equation with a sufficiently smooth test function ψ we obtain

$$-\langle u, \nabla\psi \rangle_\Omega = \langle K, \psi \rangle_\Omega - \langle g, N \cdot \psi \rangle_{\partial\Omega}. \tag{6.1.3}$$

The equations (6.1.2) and (6.1.3) can be used for the definition of very weak solutions. We go one step further and consider each right hand side of (6.1.2) and (6.1.3) as one functional in ϕ, ψ, respectively.

For a good formulation, we need to define some spaces of functions and functionals. Thus for $w \in A_q$ we consider the following spaces of functions and functionals:

$$\begin{aligned} Y^{2,q'}_{w'}(\Omega) &:= \{u \in W^{2,q'}_{w'}(\Omega) \mid u|_{\partial\Omega} = 0\}, \\ Y^{-2,q}_{w}(\Omega) &:= (Y^{2,q'}_{w'}(\Omega))' \quad \text{and} \\ W^{-1,q}_{w,0}(\Omega) &= (W^{1,q'}_{w'}(\Omega))'. \end{aligned} \tag{6.1.4}$$

Then for suitable F, K and g one obtains for the right hand sides of (6.1.2) and (6.1.3)

$$[\phi \mapsto \langle F, \phi\rangle_\Omega - \langle g, N\cdot\nabla\phi\rangle_{\partial\Omega}] \in Y_w^{-2,q}(\Omega)$$
$$[\psi \mapsto \langle K, \psi\rangle_\Omega - \langle g, N\cdot\psi\rangle_{\partial\Omega}] \in W_{w,0}^{-1,q}(\Omega).$$

Moreover, we define the divergence-free version

$$Y_{w',\sigma}^{2,q'}(\Omega) := \{\phi \in Y_{w'}^{2,q'}(\Omega) \mid \operatorname{div}\phi = 0\} \quad\text{and}\quad Y_{w,\sigma}^{-2,q}(\Omega) := (Y_{w',\sigma}^{2,q'}(\Omega))'. \tag{6.1.5}$$

We consider external forces $f \in Y_w^{-2,q}(\Omega)$ and divergences $k \in W_{w,0}^{-1,q}(\Omega)$.

Lemma 6.1.1. *$C^\infty(\overline{\Omega})$ is dense in $Y_w^{-2,q}(\Omega)$ and in $W_{w,0}^{-1,q}(\Omega)$.*

Proof. $Y_{w'}^{2,q'}(\Omega)$ is reflexive being a closed subspace of the reflexive space $W_{w'}^{2,q'}(\Omega)$. Let $x \in Y_w^{-2,q}(\Omega)' = Y_{w'}^{2,q'}(\Omega)$ such that $\langle\phi, x\rangle = 0$ for all $\phi \in C^\infty(\overline{\Omega})$. This yields $x = 0$ and the assertion is proved. The assertion about $W_{w,0}^{-1,q}(\Omega)$ is proved in the same way. $\square$

Note that these spaces do not consist of distributions on Ω since $C_0^\infty(\Omega)$ is neither dense in $Y_{w'}^{2,q'}(\Omega)$ nor in $W_{w'}^{1,q'}(\Omega)$. This leads to some difficulties when talking about derivatives. However, restricting f or k to test functions $\phi \in C_0^\infty(\Omega)$ one obtains an element of $W_w^{-2,q}(\Omega)$ or $W_w^{-1,q}(\Omega)$, respectively. If we say that equations are fulfilled in the distributional sense, we consider these restrictions. Our space of test functions will be $Y_{w',\sigma}^{2,q'}(\Omega)$ defined in (6.1.5) which is equal to the domain of the Stokes operator in $L_{w',\sigma}^{q'}(\Omega)$. It will turn out in the proof of Theorem 6.1.4 that this is no coincidence.

Definition 6.1.2. Let $f \in Y_w^{-2,q}(\Omega)$ and $k \in W_{w,0}^{-1,q}(\Omega)$. A function $u \in L_w^q(\Omega)$ is called

1. a very weak solution to the Stokes problem with respect to the data f and k if

$$-\langle u, \Delta\phi\rangle = \langle f, \phi\rangle, \qquad \text{for all } \phi \in Y_{w',\sigma}^{2,q'}(\Omega) \text{ and} \tag{6.1.6}$$
$$-\langle u, \nabla\psi\rangle = \langle k, \psi\rangle, \qquad \text{for all } \psi \in W_{w'}^{1,q'}(\Omega). \tag{6.1.7}$$

2. a very weak solution to the Stokes resolvent problem with respect to the data f and k and $\lambda \in \mathbb{C}$, if

$$\langle\lambda u, \phi\rangle - \langle u, \Delta\phi\rangle = \langle f, \phi\rangle, \qquad \text{for all } \phi \in Y_{w',\sigma}^{2,q'}(\Omega) \text{ and} \tag{6.1.8}$$
$$-\langle u, \nabla\psi\rangle = \langle k, \psi\rangle, \qquad \text{for all } \psi \in W_{w'}^{1,q'}(\Omega). \tag{6.1.9}$$

Setting $\psi = 1$ in (6.1.7) and (6.1.9) it follows that a necessary condition for the existence of such a very weak solution u is $\langle k, 1\rangle = 0$. This condition is the analogue to the compatibility condition $\langle k, 1\rangle = \langle g, N\rangle_{\partial\Omega}$ between divergence and boundary values in the case of weak solutions.

Remark 6.1.3. Two comments about the missing boundary values:

1. For every $u \in L^q_w(\Omega)$ one has
$$[\phi \mapsto \langle u, \Delta\phi\rangle] \in Y_w^{-2,q}(\Omega) \quad \text{and} \quad [\psi \mapsto \langle u, \nabla\psi\rangle] \in W_{w,0}^{-1,q}(\Omega).$$
Thus any $u \in L^q_w(\Omega)$ appears as a very weak solution to the Stokes problem with respect to appropriate data. However, since $C_0^\infty(\Omega)$ is dense in $L^q_w(\Omega)$, it is impossible to define boundary values for arbitrary L^q_w-functions in the sense of a continuous linear operator from $L^q_w(\Omega)$ into some boundary space which coincides with the usual trace on smooth functions.

2. Dealing with very weak solutions one can define boundary values adding the term $\langle g, N \cdot \nabla\phi\rangle_{\partial\Omega}$ on the right hand side of (6.1.6) and $\langle g, N \cdot \psi\rangle_{\partial\Omega}$ on the right hand side of (6.1.7). This is done in e.g. in [3], [17] and [30] in the case of more regular data. However, one easily sees that if $g \in T_w^{0,q}(\partial\Omega)$ then
$$G = [\phi \mapsto \langle g, N \cdot \nabla\phi\rangle_{\partial\Omega}] \in Y_w^{-2,q}(\Omega) \;\text{ and }\; K = [\psi \mapsto \langle g, N \cdot \psi\rangle_{\partial\Omega}] \in W_{w,0}^{-1,q}(\Omega),$$
the spaces of external forces and divergences, respectively. This means
$$\begin{aligned} -\langle u, \Delta\phi\rangle &= \langle f, \phi\rangle + \langle g, N \cdot \nabla\phi\rangle_{\partial\Omega} = \langle f + G, \phi\rangle \;\text{ and} \\ -\langle u, \nabla\psi\rangle &= \langle k, \psi\rangle + \langle g, N \cdot \psi\rangle_{\partial\Omega} = \langle k + K, \psi\rangle. \end{aligned}$$
Hence, since the data is so irregular, it is impossible to distinguish between force or divergence and boundary value.

3. In Section 6.3 we will consider the case of more regular (distributional) forces and divergences. It will be described how to regain the possibility of prescribing boundary data. Moreover, we will discuss why the existence and uniqueness of very weak solutions in the sense of Definition 6.1.2 does not contradict the theory of strong solutions to the Stokes equations in weighted spaces established in [25], [26].

Theorem 6.1.4. *Let $f \in Y_w^{-2,q}(\Omega)$, $k \in W_{w,0}^{-1,q}(\Omega)$ with $\langle k, 1\rangle = 0$ and let $\lambda \in \Sigma_\varepsilon \cup \{0\}$ with $0 < \varepsilon < \frac{\pi}{2}$. Then there exists a unique very weak solution $u \in L^q_w(\Omega)$ to the Stokes resolvent problem in the sense of Definition 6.1.2.2. It fulfills the a priori estimate*
$$\lambda \|u|_{Y^{2,q'}_{w',\sigma}(\Omega)}\|_{Y^{-2,q}_{w,\sigma}} + \|u\|_{q,w} \leq c(\|f\|_{Y_w^{-2,q}(\Omega)} + \|k\|_{W^{-1,q}_{w,0}}) \tag{6.1.10}$$
with $c = c(\Omega, q, w, \varepsilon) > 0$ depending A_q-consistently on w.

Proof. Step 1. Let $v \in L^{q'}_{w'}(\Omega)$. By the existence of strong solutions to the Stokes resolvent problem ([26, Theorem 3.3] in the case of weighted and [29], [47] in the case of unweighted spaces) there are unique functions $\phi \in W^{2,q'}_{w'}(\Omega)$ and $\psi \in W^{1,q'}_{w'}(\Omega)$ which depend linearly on v and such that
$$\lambda\phi - \Delta\phi + \nabla\psi = v \quad \text{and} \quad \operatorname{div}\phi = 0 \quad \text{in } \Omega, \quad \phi|_{\partial\Omega} = 0 \quad \text{and} \quad \int \psi = 0. \tag{6.1.11}$$
This solution satisfies
$$\lambda\|\phi\|_{q',w'} + \|\phi\|_{2,q',w'} + \|\psi\|_{1,q',w'} \leq c\|v\|_{q',w'}$$

with an A_q-consistent constant c.
Step 2. (Existence and a priori estimates) Setting for $v \in L^{q'}_{w'}(\Omega)$

$$\langle u, v\rangle := \langle f, \phi\rangle - \langle k, \psi\rangle, \quad \text{with } (\phi, \psi) \text{ as in } (6.1.11), \tag{6.1.12}$$

we obtain

$$\begin{aligned} |\langle u, v\rangle| &\leq |\langle f, \phi\rangle| + |\langle k, \psi\rangle| \\ &\leq \|f\|_{Y^{-2,q}_w}\|\phi\|_{2,q',w'} + \|k\|_{W^{-1,q}_{w,0}}\|\psi\|_{1,q',w'} \\ &\leq c(\|f\|_{Y^{-2,q}_w} + \|k\|_{W^{-1,q}_{w,0}})\|v\|_{q',w'}. \end{aligned}$$

Thus $u \in (L^{q'}_{w'}(\Omega))' = L^q_w(\Omega)$ and fulfills $\|u\|_{q,w} \leq c(\|f\|_{Y^{-2,q}_w} + \|k\|_{W^{-1,q}_{w,0}})$ with c independent of λ and depending A_q-consistently on w.

We now show that u is a very weak solution to the Stokes problem with respect to f and k. Choose test functions $\phi \in Y^{2,q'}_{w',\sigma}(\Omega)$ and $\psi \in W^{1,q'}_{w'}(\Omega)$. Then setting $v = \lambda\phi - \Delta\phi + \nabla\psi$ we obtain from the uniqueness of strong solutions

$$\langle u, \lambda\phi - \Delta\phi + \nabla\psi\rangle = \langle u, v\rangle = \langle f, \phi\rangle - \langle k, \psi\rangle.$$

Since ϕ and ψ were chosen arbitrarily, (6.1.8) and (6.1.9) are fulfilled.

Moreover, let $\phi \in Y^{2,q'}_{w',\sigma}(\Omega)$. Then we obtain

$$\begin{aligned} |\langle \lambda u, \phi\rangle| &\leq |\langle u, \Delta\phi\rangle| + |\langle f, \phi\rangle| \leq (\|u\|_{q,w} + \|f\|_{Y^{-2,q}_w})\|\phi\|_{2,q',w'} \\ &\leq c(\|f\|_{Y^{-2,q}_w} + \|k\|_{W^{-1,q}_{w,0}(\Omega)})\|\phi\|_{2,q',w'}. \end{aligned}$$

Combining this with the previous estimate we get (6.1.10),
Step 3. (Uniqueness) Assume $U \in L^q_w(\Omega)$ is a very weak solution to the Stokes resolvent problem with respect to f and k. As above for every $v \in L^{q'}_{w'}(\Omega)$ we find $\phi \in Y^{2,q'}_{w',\sigma}(\Omega)$ and $\psi \in W^{1,q'}_{w'}(\Omega)$ such that $\lambda u - \Delta\phi + \nabla\psi = v$. If we add the equations (6.1.8) and (6.1.9) we obtain

$$\langle U, v\rangle = \langle U, \lambda\phi - \Delta\phi + \nabla\psi\rangle = \langle f, \phi\rangle - \langle k, \psi\rangle = \langle u, v\rangle.$$

Since $v \in L^{q'}_{w'}(\Omega)$ was arbitrary, we obtain $u = U$. $\square$

Theorem 6.1.5. *Let f and k be chosen as in Theorem 6.1.4 and let $u \in L^q_w(\Omega)$ be the associated very weak solution to the Stokes problem. Then there exists a unique pressure functional $p \in W^{-1,q}_{w,0}(\Omega)$ (unique modulo constants) such that (u, p) solves*

$$-\langle u, \Delta\phi\rangle - \langle p, \operatorname{div}\phi\rangle = \langle f, \phi\rangle \quad \textit{for all } \phi \in Y^{2,q'}_{w'}(\Omega).$$

In particular

$$-\Delta u + \nabla p|_{C^\infty_0(\Omega)} = f|_{C^\infty_0(\Omega)}$$

in the sense of distributions. The functions (u, p) fulfill the inequality

$$\|u\|_{q,w} + \|p\|_{W^{-1,q}_{w,0}} \leq c\left(\|f\|_{Y^{-2,q}_w} + \|k\|_{W^{-1,q}_{w,0}}\right), \tag{6.1.13}$$

where $c = c(\Omega, q, w) > 0$.

Proof. By Lemma 6.1.1 there exist sequences $(f_n)_n, (k_n)_n \subset C^\infty(\overline{\Omega})$ such that

$$f_n \xrightarrow{Y_w^{-2,q}(\Omega)} f \quad \text{and} \quad k_n \xrightarrow{W_{w,0}^{-1,q}(\Omega)} k.$$

Then by [26, Theorem 3.3] there exist unique solutions $(u_n, p_n) \in W_w^{2,q}(\Omega) \times W_w^{1,q}(\Omega)$ such that

$$-\Delta u_n + \nabla p_n = f_n, \quad \operatorname{div} u_n = k_n, \quad u_n|_{\partial\Omega} = 0, \quad \int p_n = 0.$$

Integration by parts implies that u_n is a very weak solution with respect to f_n, k_n. Now the a priori estimate (6.1.2) shows $u_n \xrightarrow{L_w^q(\Omega)} u$. For $\phi \in W_{w'}^{1,q'}(\Omega)$ with $\int \phi = 0$ let $\zeta \in Y_{w'}^{2,q'}(\Omega)$ be the solution to $-\Delta\zeta + \nabla\pi = 0$ and $\operatorname{div}\zeta = \phi$. Then $\|\zeta\|_{2,q',w'} \leq c\|\phi\|_{1,q',w'}$. Thus we obtain

$$\begin{aligned}
|\langle p_n - p_m, \phi\rangle| &= |\langle p_n - p_m, \operatorname{div}\zeta\rangle| = |\langle \nabla(p_n - p_m), \zeta\rangle| \\
&\leq |\langle \Delta(u_n - u_m), \zeta\rangle| + |\langle f_n - f_m, \zeta\rangle| \\
&\leq c(\|u_n - u_m\|_{q,w} + \|f_n - f_m\|_{Y_w^{-2,q}})\|\zeta\|_{2,q',w'} \\
&\leq c(\|u_n - u_m\|_{q,w} + \|f_n - f_m\|_{Y_w^{-2,q}})\|\phi\|_{1,q',w'}.
\end{aligned}$$

Thus $\|p_n - p_m\|_{-1,q,w,0} \leq c(\|u_n - u_m\|_{q,w} + \|f_n - f_m\|_{Y_w^{-2,q}}) \xrightarrow{n,m\to\infty} 0$ and $(p_n)_n$ is a Cauchy sequence converging to some $p \in W_{w,0}^{-1,q}(\Omega)$. For this p

$$-\langle u, \Delta\phi\rangle - \langle p, \operatorname{div}\phi\rangle = \lim_{n\to\infty}(-\langle u_n, \Delta\phi\rangle - \langle p_n, \operatorname{div}\phi\rangle) = \lim_{n\to\infty}\langle f_n, \phi\rangle = \langle f, \phi\rangle$$

holds for every $\phi \in Y_{w'}^{2,q'}(\Omega)$. The estimate (6.1.13) follows from the estimates for p_n and u_n. □

6.2 Regularity

The following theorem describes how strong solutions fit into the context of very weak solutions considered in the previous section. Moreover, it prepares further considerations about boundary values in the case of low regularity data.

Lemma 6.2.1. *Let $1 < q, r < \infty$, $w \in A_q$ and $\tilde{w} \in A_r$ such that*

$$W_{w'}^{1,q'}(\Omega) \hookrightarrow L_{\tilde{w}'}^{r'}(\Omega) \hookrightarrow L_{w'}^{q'}(\Omega). \tag{6.2.1}$$

Then

$$L_{\tilde{w}}^r(\Omega) \hookrightarrow W_{w,0}^{-1,q}(\Omega) \quad \text{and} \quad W_{\tilde{w}}^{-1,r}(\Omega) \hookrightarrow Y_w^{-2,q}(\Omega)$$

Proof. The first assertion follows immediately from (6.2.1) by duality. For the second assertion we estimate

$$\|v\|_{1,r',\tilde{w}'} = \|\nabla v\|_{r',\tilde{w}'} + \|v\|_{r',\tilde{w}'} \leq c\,(\|\nabla v\|_{1,q',w'} + \|v\|_{1,q',w'}) \leq c\|v\|_{2,q',w'}.$$

This proves the embedding $Y_{w'}^{2,q'}(\Omega) \hookrightarrow W_{\tilde{w}',0}^{1,r'}(\Omega)$ using $v|_{\partial\Omega} = 0$ for every $v \in Y_{w'}^{2,q'}(\Omega)$. Thus again the assertion follows from (6.2.1) by duality. □

The reason why we require these embeddings is that Sobolev-like inequalities in weighted spaces need strong assumptions on the weight-functions. In [28] sufficient conditions for such embeddings are proved using the continuity of singular integral operators shown in [41]. See Section 10.1 for further considerations concerning this problem.

Theorem 6.2.2. *Assume that $f \in Y_w^{-2,q}(\Omega)$ and $k \in W_{w,0}^{-1,q}(\Omega)$ allow a decomposition into*

$$\begin{aligned} \langle f, \phi\rangle &= \langle F, \phi\rangle - \langle g, N\cdot\nabla\phi\rangle_{\partial\Omega} \qquad &&\text{for all } \phi \in Y_{w'}^{2,q'}(\Omega), \\ \langle k, \psi\rangle &= \langle K, \psi\rangle - \langle g, N\cdot\psi\rangle_{\partial\Omega} \qquad &&\text{for all } \psi \in W_{w'}^{1,q'}(\Omega) \end{aligned} \tag{6.2.2}$$

with $g \in T_w^{0,q}(\partial\Omega), F \in W_{\tilde{w}}^{-1,r}(\Omega), K \in L_{\tilde{w}}^r(\Omega)$, where $1 < r < \infty$ and $\tilde{w} \in A_r$ are chosen according to (6.2.1). Then one has:

1. *Such a decomposition is uniquely defined by f and k.*
2. *For $\lambda \in \Sigma_\varepsilon \cup \{0\}$ every strong solution u to the Stokes resolvent problem corresponding to the data $g \in T_w^{2,q}(\partial\Omega)$, $F \in L_w^q(\Omega)$ and $K \in W_w^{1,q}(\Omega)$ is a very weak solution corresponding to the data f and k with the notation of (6.2.2).*
3. *If $\lambda \in \Sigma_\varepsilon \cup \{0\}$, $g \in T_w^{2,q}(\partial\Omega)$, $F \in L_w^q(\Omega)$ and $K \in W_w^{1,q}(\Omega)$ with $\int_\Omega K = \int_{\partial\Omega} N\cdot g$, then the very weak solution u to the Stokes resolvent problem with respect to f and k is a strong solution with respect to F, K and g. In particular $u \in W_w^{2,q}(\Omega)$ and*

$$|\lambda|\|u\|_{q,w} + \|u\|_{2,q,w} \le c(\|F\|_{q,w} + \|K\|_{1,q,w} + \|\lambda K\|_{W_{w,0}^{-1,q}} + \|g\|_{T_w^{2,q}} + \|\lambda g\|_{T_w^{0,q}}). \tag{6.2.3}$$

Proof. 1. Let $\langle f, \phi\rangle = \langle F_i, \phi\rangle - \langle g_i, N\cdot\nabla\phi\rangle_{\partial\Omega}$ for $i = 1, 2$ with F_i, g_i as in the assumption. Then

$$\langle F_1 - F_2, \phi\rangle = \langle g_1 - g_2, N\cdot\nabla\phi\rangle_{\partial\Omega} \quad \text{for } \phi \in Y_{w'}^{2,q'}(\Omega).$$

The latter functional vanishes on $C_0^\infty(\Omega)$ and since $F_1 - F_2$ is a distribution on Ω, it follows that $F_1 - F_2 = 0$ and hence $\langle g_1 - g_2, N\cdot\nabla\phi\rangle = 0$ for every $\phi \in Y_{w'}^{2,q'}(\Omega)$. By Theorem 5.2.1 the mapping

$$\phi \mapsto N\cdot\nabla\phi : \; Y_{w'}^{2,q'}(\Omega) \to T_{w'}^{1,q'}(\partial\Omega)$$

is surjective, hence $g_1 = g_2$. Analogously one shows that the decomposition of the divergence k is unique.
2. If $u \in W_w^{2,q}(\Omega)$ is the strong solution corresponding to the data F, K, g, then Green's formula yields for every $\phi \in Y_{w',\sigma}^{2,q'}(\Omega)$

$$\begin{aligned} \langle\lambda u, \phi\rangle - \langle u, \Delta\phi\rangle &= \langle\lambda u - \Delta u, \phi\rangle - \langle g, N\cdot\nabla\phi\rangle_{\partial\Omega} \\ &= \langle F, \phi\rangle - \langle g, N\cdot\nabla\phi\rangle_{\partial\Omega} = \langle f, \phi\rangle \end{aligned}$$

and for every $\psi \in W^{1,q}(\Omega)$

$$-\langle u, \nabla\psi\rangle = \langle K, \psi\rangle - \langle g, N\cdot\psi\rangle = \langle k, \psi\rangle.$$

Thus u is a very weak solution.
3. By Theorem 5.2.1 there exists $v_1 \in W_w^{2,q}(\Omega)$ with $v_1|_{\partial\Omega} = g$ and $\|v_1\|_{2,q,w} \le c\|g\|_{T_w^{2,q}}$ and one has

$$\langle K - \operatorname{div} v_1, 1\rangle = \langle K, 1\rangle - \langle g, N\rangle_{\partial\Omega} = 0.$$

Hence, by [26, Theorem 3.3] there exists a strong solution $v_2 \in Y_w^{2,q}(\Omega)$ with respect to the exterior force $F - \lambda v_1 + \Delta v_1$ and divergence $K - \operatorname{div} v_1$. It fulfills the estimate

$$\begin{aligned}
&|\lambda|\|v_2\|_{q,w} + \|v_2\|_{2,q,w} \\
&\leq c(\|F\|_{q,w} + \|\Delta v_1\|_{q,w} + |\lambda|\|v_1\|_{q,w} + \|K - \operatorname{div} v_1\|_{1,q,w} + |\lambda|\|K - \operatorname{div} v_1\|_{W_{w,0}^{-1,q}}) \\
&\leq c(\|F\|_{q,w} + |\lambda|\|v_1\|_{q,w} + \|K\|_{1,q,w} + |\lambda|\|K - \operatorname{div} v_1\|_{W_{w,0}^{-1,q}} + \|g\|_{T_w^{2,q}}).
\end{aligned} \tag{6.2.4}$$

Then $u = v_1 + v_2$ is a strong solution to the Stokes resolvent problem with respect to the given data. Moreover, in the case $\lambda = 0$, also the estimate is proved.

Now we repeat the above arguments with v_1 replaced by the solution to the Stokes problem

$$-\Delta v_1 + \nabla p = 0, \quad \operatorname{div} v_1 = 0 \text{ and } v_1|_{\partial\Omega} = g.$$

Then v_1 fulfills the estimate $\|v_1\|_{2,q,w} \leq c\|g\|_{T_w^{2,q}(\partial\Omega)}$. In addition, by 2. we know that v_1 is also a very weak solution with respect to the data

$$\tilde{f} = [\phi \mapsto \langle g, N \cdot \nabla\phi\rangle] \text{ and } \tilde{k} = [\psi \mapsto \langle g, N \cdot \psi\rangle].$$

Thus we obtain the estimate

$$\|v_1\|_{q,w} \leq c(\|\tilde{f}\|_{Y_w^{-2,q}} + \|\tilde{k}\|_{W_{w,0}^{-1,q}}) \leq c\|g\|_{T_w^{0,q}}.$$

Inserting this in (6.2.4) we obtain

$$\begin{aligned}
|\lambda|\|u\|_{q,w} + \|u\|_{2,q,w} \leq &|\lambda|\|v_1\|_{q,w} + \|v_2\|_{2,q,w} + |\lambda|\|v_2\|_{q,w} + \|v_2\|_{2,q,w} \\
\leq &c(\|F\|_{q,w} + \|K\|_{1,q,w} + |\lambda|\|K\|_{W_{w,0}^{-1,q}} + \|g\|_{T_w^{2,q}} + |\lambda|\|g\|_{T_w^{0,q}}).
\end{aligned}$$

Thus there exists a strong solution to the Stokes resolvent problem with respect to the given data which fulfills the estimate.

The uniqueness of very weak solutions proved in Theorem 6.1.4 together with 2. yields that u coincides with the very weak solution. In particular the very weak solution is regular according to the data. □

Remark 6.2.3. If there exist decompositions for the data f and k as in (6.2.2) even with smooth functions F, K, g this does not mean that f and k are smooth. The reason is that if $g \neq 0$, then $\phi \mapsto \langle g, N \cdot \nabla\phi\rangle$ can never be a function since it is a functional supported by the boundary and depending on derivatives.

Vice versa, if f and k are regular, e.g. $f \in W_w^{-1,q}(\Omega)$ and $k \in L_w^q(\Omega)$ allowing a decomposition according to (6.2.2), then we automatically obtain $g = 0$, which means that the very weak solution with respect to f and k has zero boundary values.

6.3 Boundary Values in the Case of More Regular Data

Our next aim is to define boundary values for very weak solutions to the Stokes problem presumed the data is sufficiently regular. To this aim we find a Banach space containing all solutions corresponding to such data and a continuous linear operator on this space coinciding with the usual trace on $C^\infty(\overline{\Omega})$.

From now on let $1 < r < \infty$, $\tilde{w} \in A_r$ such that (6.2.1) is fulfilled and take $F \in W^{-1,r}_{\tilde{w}}(\Omega)$ and $K \in L^r_{\tilde{w}}(\Omega)$ and $g \in T^{0,q}_w(\partial\Omega)$. Then

$$\begin{aligned}&[\phi \mapsto \langle F, \phi\rangle - \langle g, N \cdot \nabla\phi\rangle_{\partial\Omega}] \in Y^{-2,q}_w(\Omega) \text{ and}\\ &[\psi \mapsto \langle K, \psi\rangle - \langle g, N \cdot \psi\rangle_{\partial\Omega}] \in W^{-1,q}_{w,0}(\Omega).\end{aligned}$$

Thus by Theorem 6.1.4 there exists a unique function $u \in L^q_w(\Omega)$ such that

$$\begin{aligned}-\langle u, \Delta\phi\rangle &= \langle F, \phi\rangle - \langle g, N \cdot \nabla\phi\rangle_{\partial\Omega} \quad \forall \phi \in Y^{2,q'}_{w',\sigma}(\Omega) \text{ and}\\ -\langle u, \nabla\psi\rangle &= \langle K, \psi\rangle - \langle g, N \cdot \psi\rangle_{\partial\Omega} \quad \forall \psi \in W^{1,q'}_{w}(\Omega).\end{aligned}$$

However, the question arises in which sense this solution u fulfills $u|_{\partial\Omega} = g$.

As a large space of functions in which the definition of tangential boundary conditions is possible we define

$$\begin{aligned}\tilde{W}^{q,r}_{w,\tilde{w}}(\Omega) :=& \big\{u \in L^q_w(\Omega) \;\big|\; (\Delta u)|_{C^\infty_{0,\sigma}(\Omega)} \text{ extends to an element of } (W^{1,r'}_{\tilde{w}',0,\sigma}(\Omega))'\big\}\\ =& \big\{u \in L^q_w(\Omega) \;\big|\; \exists c > 0,\ |\langle u, \Delta\phi\rangle| \le c\|\phi\|_{1,r',\tilde{w}'}\ \forall \phi \in C^\infty_{0,\sigma}(\Omega)\big\}.\end{aligned} \tag{6.3.1}$$

We will omit the symbol Ω and write $\tilde{W}^{q,r}_{w,\tilde{w}}$ if no confusion can occur.

To guarantee that the extension in (6.3.1) is uniquely defined by the values of $\langle u, \Delta\phi\rangle$ for $\phi \in C^\infty_{0,\sigma}(\Omega)$ we use the following Lemma.

Lemma 6.3.1. *Let $r' > 1$, $\tilde{w} \in A_{r'}$ and $k \in \mathbb{N}$. Then one has*

$$\overline{C^\infty_{0,\sigma}(\Omega)}^{W^{k,r'}_{\tilde{w}'}(\Omega)} = W^{k,r'}_{\tilde{w}',0,\sigma}(\Omega).$$

Proof. We have to prove the density $C^\infty_{0,\sigma}(\Omega) \hookrightarrow W^{k,r'}_{\tilde{w}',0,\sigma}(\Omega)$. To do this let

$$v \in (W^{k,r'}_{\tilde{w}',0,\sigma}(\Omega))', \quad \langle v, \phi\rangle = 0 \;\text{ for all }\; \phi \in C^\infty_{0,\sigma}(\Omega).$$

By the Hahn-Banach theorem v extends to an element $V \in W^{-k,r}_{\tilde{w}}(\Omega)$. Since $\langle V, \phi\rangle = 0$ for every $\phi \in C^\infty_{0,\sigma}(\Omega)$, it follows by de Rham's theorem [51] that $V = \nabla U$ for some $U \in C^\infty_0(\Omega)'$. By Theorem 4.3.1 there exists for every $\phi \in C^\infty_0(\Omega)$ with $\int_\Omega \phi = 0$ some $\zeta \in C^\infty_0(\Omega)$ with $\operatorname{div}\zeta = \phi$ and $\|\zeta\|_{k,r',\tilde{w}'} \le c\|\phi\|_{k-1,r',\tilde{w}'}$. Thus we can estimate

$$|\langle U, \phi\rangle| = |\langle U, \operatorname{div}\zeta\rangle| = |\langle \nabla U, \zeta\rangle| \le c\|V\|_{-k,r,\tilde{w}}\|\phi\|_{k-1,r',\tilde{w}'}$$

for every ϕ with $\int_\Omega \phi = 0$. This proves $U \in W^{1-k,r}_{\tilde{w}}(\Omega)$ and we obtain for every $\psi \in W^{k,r'}_{\tilde{w}',0,\sigma}(\Omega)$ using the definition of the distributional derivative and the fact that we can approximate ψ by $C^\infty_0(\Omega)$-functions in the norm of $W^{k,r'}_{\tilde{w}',0}(\Omega)$

$$\langle v, \psi\rangle = \langle V, \psi\rangle = \langle \nabla U, \psi\rangle = -\langle U, \operatorname{div}\psi\rangle = 0.$$

Now the Hahn-Banach Theorem proves the assertion. $\square$

Lemma 6.3.2. *$\tilde{W}^{q,r}_{w,\tilde{w}}$ is a Banach space equipped with the norm*

$$\|u\|_{\tilde{W}^{q,r}_{w,\tilde{w}}} = \|u\|_{q,w} + \|\Delta u|_{C^\infty_{0,\sigma}(\Omega)}\|_{(W^{1,r'}_{\tilde{w}',0,\sigma}(\Omega))'}.$$

Proof. Let $(u_n)_n$ be a Cauchy sequence in $W^{q,r}_{w,\tilde{w}}$. Then There exists $u \in L^q_w(\Omega)$ and $v \in (W^{1,r'}_{\tilde{w},0,\sigma}(\Omega))'$.

$$u_n \xrightarrow{L^q_w(\Omega)} u \quad \text{and} \quad \Delta u_n \xrightarrow{(W^{1,r'}_{\tilde{w},0,\sigma}(\Omega))'} v.$$

It remains to show that $\langle v, \phi\rangle = \langle \Delta u, \phi\rangle$ for all $\phi \in C^\infty_{0,\sigma}(\Omega)$. From the continuity of $\Delta : L^q_w(\Omega) \to W^{-2,q}_w(\Omega)$ we have

$$\langle \Delta u, \phi\rangle \xleftarrow{n\to\infty} \langle \Delta u_n, \phi\rangle \xrightarrow{n\to\infty} \langle v, \phi\rangle$$

for every $\phi \in C^\infty_{0,\sigma}(\Omega)$. □

Lemma 6.3.3. *Let $f \in Y^{-2,q}_{w,\sigma}(\Omega)$ with*

$$\langle f, \phi\rangle = 0 \quad \textit{for every} \quad \phi \in C^\infty_{0,\sigma}(\Omega).$$

Then there exists an extension $F \in Y^{-2,q}_w(\Omega)$ such that $\langle F, \phi\rangle = 0$ for every $\phi \in C^\infty_0(\Omega)$ and with $\|F\|_{Y^{-2,q}_w} \le c\|f\|_{Y^{-2,q}_{w,\sigma}}$.

Proof. First we show that $\tilde{f}$ defined by

$$\langle \tilde{f}, \phi\rangle = \begin{cases} \langle f, \phi\rangle & \text{if } \phi \in Y^{2,q'}_{w',\sigma}(\Omega) \\ 0 & \text{if } \phi \in W^{2,q'}_{w',0}(\Omega) \end{cases}$$

is a continuous functional on $Y^{2,q'}_{w',\sigma}(\Omega) + W^{2,q'}_{w',0}(\Omega)$. Well-definedness and linearity are clear since $\langle f, \phi\rangle = 0$ on $Y^{2,q'}_{w',\sigma}(\Omega) \cap W^{2,q'}_{w',0}(\Omega) = W^{2,q'}_{w',0,\sigma}(\Omega) = \overline{C^\infty_{0,\sigma}(\Omega)}^{2,q',w'}$ by Lemma 6.3.1.

Thus it remains to prove continuity. By Theorem 4.3.1 there exists a continuous linear operator $T : \{v \in W^{1,q'}_{w',0}(\Omega) \mid \int v = 0\} \to W^{2,q'}_{w',0}(\Omega)$ such that $\operatorname{div}(Tv) = v$.

Let $\phi \in Y^{2,q'}_{w',\sigma}(\Omega) + W^{2,q'}_{w',0}(\Omega)$. Then

$$\operatorname{div}\phi \in W^{1,q'}_{w',0}(\Omega), \quad \int \operatorname{div}\phi = \int_{\partial\Omega} \phi\cdot N = 0$$

and we obtain

$$|\langle \tilde{f}, \phi\rangle| = |\langle \tilde{f}, \phi - T(\operatorname{div}\phi)\rangle + \langle \tilde{f}, T(\operatorname{div}\phi)\rangle| = |\langle f, \phi - T(\operatorname{div}\phi)\rangle| \le c\|f\|_{Y^{-2,q}_{w,\sigma}}\|\phi\|_{2,q',w'}$$

and consequently

$$\|\tilde{f}\|_{(Y^{2,q'}_{w',\sigma}(\Omega)+W^{2,q'}_{w',0}(\Omega))'} \le c\|f\|_{Y^{-2,q}_{w,\sigma}}.$$

By the Hahn-Banach Theorem we may extend $\tilde{f}$ to an element $F \in Y^{-2,q}_w(\Omega)$ with $\|F\|_{Y^{-2,q}_w} = \|\tilde{f}\|_{(Y^{2,q'}_{w',\sigma}(\Omega)+W^{2,q'}_{w',0}(\Omega))'}$. This finishes the proof. □

The following Lemma is crucial when proving the well-definedness of the tangential component of the trace on $\tilde{W}^{q,r}_{w,\tilde{w}}$.

Lemma 6.3.4. *$C^\infty(\overline{\Omega})$ is dense in $\tilde{W}^{q,r}_{w,\tilde{w}}$.*

Proof. Let $u \in \tilde{W}^{q,r}_{w,\tilde{w}}$. Then by definition and Lemma 6.2.1 we have

$$\Delta u|_{C^\infty_{0,\sigma}} \in (W^{1,r'}_{\tilde{w}',0,\sigma}(\Omega))' \hookrightarrow Y^{-2,q}_{w,\sigma}(\Omega).$$

The Hahn-Banach theorem yields the existence of some

$$f \in (W^{1,r'}_{\tilde{w}',0}(\Omega))' = W^{-1,r}_{\tilde{w}}(\Omega) \subset Y^{-2,q}_{w}(\Omega)$$

such that

$$\langle f, \phi\rangle = \langle \Delta u, \phi\rangle \quad \text{for all} \quad \phi \in C^\infty_{0,\sigma}(\Omega).$$

By Lemma 6.3.3 there exists an extension $F \in Y^{-2,q}_w(\Omega)$ of $(\langle u, \Delta\cdot\rangle - f)|_{Y^{2,q'}_{w',\sigma}(\Omega)}$ vanishing on $C^\infty_0(\Omega)$. By Theorem 4.1.5 there exists a $v \in L^q_w(\Omega)$ such that

$$\langle v, \Delta\phi\rangle = \langle F, \phi\rangle \quad \text{for all} \quad \phi \in Y^{2,q'}_{w'}(\Omega).$$

This v is harmonic on Ω because $\langle F, \phi\rangle = 0$ for all $\phi \in C^\infty_0(\Omega)$.

Now we assume temporarily that Ω is star-shaped with respect to some ball $B_r(0)$ with center 0 and radius r. So we may set $v_\lambda(x) := v(\lambda x)$, where $\lambda \in (0,1)$ and $v_n(x) := v_{\lambda_n}(x)$, where $(\lambda_n) \subset (0,1)$ is a sequence converging to 1. Then by Theorem 4.3.1 we have $v_n \xrightarrow{n\to\infty} v$ in $L^q_w(\Omega)$. Moreover, since every v_n is harmonic we have $\Delta v_n - \Delta v = 0$ for all n which yields the convergence in $\tilde{W}^{q,r}_{w,\tilde{w}}$.

Now let Ω be an arbitrary bounded $C^{1,1}$-domain. Then $\Omega = \bigcup_{i=1}^N \Omega_i$ with strictly star-shaped domains Ω_i. Let $(\alpha_i)_i$ be a partition of unity subordinate to this covering. For $i = 1, ..., N$ let $(v^{(i)}_n)_n$ be the sequences of harmonic functions constructed above converging to $v^{(i)} := v|_{\Omega_i}$ in $\tilde{W}^{q,r}_{w,\tilde{w}}(\Omega_i)$. We show that

$$v_n := \sum_{j=1}^N \alpha_j v^{(j)}_n \xrightarrow{n\to\infty} v \text{ in } \tilde{W}^{q,r}_{w,\tilde{w}}(\Omega).$$

Convergence in $L^q_w(\Omega)$ is clear since multiplication with functions in $C^\infty_0(\overline{\Omega})$ is continuous. Moreover,

$$\begin{aligned}
\Delta\left(\sum_{j=1}^N \alpha_j v^{(j)}_n\right) &= \sum_{j=1}^N((\Delta\alpha_j)v^{(j)}_n + 2\nabla\alpha_j\nabla v^{(j)}_n + 0) \\
&\xrightarrow{W^{-1,r}_{\tilde{w}}(\Omega)} \sum_{j=1}^N((\Delta\alpha_j)v + 2\nabla\alpha_j\nabla v + \alpha_j\Delta v) \\
&= \Delta\left(\sum_{j=1}^N \alpha_j v\right) = 0.
\end{aligned}$$

The convergence holds because

$$(\Delta\alpha_j)v_n^{(j)} \xrightarrow{n\to\infty} (\Delta\alpha_j)v^{(j)} = (\Delta\alpha_j)v \;\text{ in }\; L^q_w(\Omega) \;\text{ and }\; L^q_w(\Omega) \hookrightarrow W^{-1,r}_{\tilde{w}}(\Omega)$$

and

$$2\nabla\alpha_j\nabla v_n^{(j)} \xrightarrow{n\to\infty} 2\nabla\alpha_j\nabla v^{(j)} \;\text{ in }\; W^{-1,q}_w(\Omega) \;\text{ and }\; W^{-1,q}_w(\Omega) \hookrightarrow W^{-1,r}_{\tilde{w}}(\Omega).$$

Moreover, we have

$$\begin{aligned}
\langle u-v, \Delta\phi\rangle &= \langle f,\phi\rangle + \langle F,\phi\rangle - \langle F,\phi\rangle = \langle f,\phi\rangle \quad && \text{for } \phi \in Y^{2,q'}_{w',\sigma}(\Omega)\\
\langle u-v, \nabla\psi\rangle &=: \langle k,\psi\rangle && \text{for } \psi \in W^{1,q'}_{w'}(\Omega).
\end{aligned}$$

Let $(f_n)_n, (k_n)_n \subset C^\infty(\overline{\Omega})$ such that $f_n \xrightarrow{n\to\infty} f$ in $W^{-1,r}_{\tilde{w}}(\Omega)$ and $k_n \xrightarrow{n\to\infty} k$ in $W^{-1,q}_{w,0}(\Omega)$. The embedding $W^{-1,r}_{\tilde{w}}(\Omega) \hookrightarrow Y^{-2,q}_w(\Omega)$ and the a priori estimate for very weak solutions to the Stokes equations (6.1.10) yields that the sequence of very weak solutions $(u_n)_n$ to the Stokes problem with respect to f_n and k_n converges to $u-v$ in $L^q_w(\Omega)$. By the regularity of the data and of the boundary (Theorem 6.2.2) one has $u_n \in W^{2,q}_w(\Omega)$.

We show that u_n tends to $u-v$ in $\tilde{W}^{q,r}_{w,\tilde{w}}$. The convergence in $L^q_w(\Omega)$ is already shown. Moreover for $\phi \in C^\infty_{0,\sigma}(\Omega)$ one has

$$\sup_{\phi\in C^\infty_{0,\sigma}, \|\phi\|_{1,r',\tilde{w}'}=1} |\langle u_n, \Delta\phi\rangle - \langle f,\phi\rangle| = \sup_{\phi\in C^\infty_{0,\sigma}, \|\phi\|_{1,r',\tilde{w}'}=1} |\langle f_n,\phi\rangle - \langle f,\phi\rangle| \xrightarrow{n\to\infty} 0.$$

Thus the sequence $(u_n+v_n)_n \subset W^{2,q}_w(\Omega)$ approximates u in the norm of $\tilde{W}^{q,r}_{w,\tilde{w}}$. Since $C^\infty(\overline{\Omega})$ is dense in $W^{2,q}_w(\Omega)$, the assertion is proved. □

It is not difficult to see that if $\phi \in W^{2,q}_w(\Omega)$ with $\phi|_{\partial\Omega} = 0$ and $\operatorname{div}\phi = 0$, then $N\cdot\nabla\phi$ is purely tangential. The next Lemma shows that vice versa every purely tangential function on the boundary is a normal derivative of such a function. This ensures that the set of test functions is sufficiently large.

Lemma 6.3.5. *Let Ω be a bounded $C^{1,1}$-domain, $1 < q < \infty$ and $w \in A_q$. For every $h \in T^{1,q}_w(\partial\Omega)$ with $N\cdot h = 0$ there exists a function $\phi_h \in W^{2,q}_w(\Omega)$ such that*

$$\phi_h|_{\partial\Omega} = 0,\; N\cdot\nabla\phi_h = h \textit{ and } \operatorname{div}\phi_h = 0.$$

Moreover ϕ_h can be chosen depending linearly on h and fulfilling the estimate

$$\|\phi_h\|_{2,q,w} \le c\|h\|_{T^{1,q}_w(\partial\Omega)}$$

with a constant $c = c(\Omega,q,w) > 0$.

Proof. For $h \in T^{1,q}_w(\partial\Omega)$ there exists by Theorem 5.2.1 a function $\psi_h \in W^{2,q}_w(\Omega)$ depending linearly on h such that

$$\psi_h|_{\partial\Omega} = 0,\; N\cdot\nabla\psi_h = h \text{ and } \|\psi_h\|_{2,q,w} \le c\|h\|_{T^{1,q}_w(\partial\Omega)}.$$

Since in addition $h = N\cdot\nabla\psi_h$ is purely tangential, one can show (see [30]) that $\operatorname{div}\psi_h \in W^{1,q}_{w,0}(\Omega)$. Thus by Theorem 4.3.1 there exists a function $\zeta \in W^{2,q}_{w,0}(\Omega)$ with $\operatorname{div}\zeta = \operatorname{div}\psi_h$, depending linearly on ψ_h and satisfying the estimate $\|\zeta\|_{2,q,w} \le c\|\operatorname{div}\psi_h\|_{1,q,w} \le c\|\psi_h\|_{2,q,w}$.

Now $\phi_h := \psi_h - \zeta$ solves the problem. □

Using this lemma we define the tangential component of $u \in \tilde{W}^{q,r}_{w,\tilde{w}}$ on the boundary as follows. If $u \in \tilde{W}^{q,r}_{w,\tilde{w}}$ and $\phi \in Y^{2,q'}_{w',\sigma}(\Omega)$ we use the notation

$$\langle \Delta_\sigma u, \phi \rangle := \lim_{n\to\infty} \langle u, \Delta\phi_n \rangle \tag{6.3.2}$$

where $(\phi_n)_n \in C^\infty_{0,\sigma}(\Omega)$ converges to ϕ in $W^{1,r'}_{\tilde{w}',0,\sigma}(\Omega)$. This is possible by Lemma 6.3.1, and by the definition of $\tilde{W}^{q,r}_{w,\tilde{w}}$ the functional $\Delta_\sigma u$ is independent of the approximation (ϕ_n).

Theorem 6.3.6. *There exists a continuous linear operator*

$$\begin{array}{lll} \gamma : \tilde{W}^{q,r}_{w,\tilde{w}} & \to T^{0,q}_w(\partial\Omega), & \textit{such that} \\ \langle \gamma(u), h \rangle_{\partial\Omega} & = \langle u, \Delta\phi_h \rangle - \langle \Delta_\sigma u, \phi_h \rangle & \textit{if } N\cdot h = 0, \\ \langle \gamma(u), h \rangle_{\partial\Omega} & = 0 & \textit{if } h = \tilde{h}N \end{array} \tag{6.3.3}$$

for $h \in T^{1,q'}_{w'}(\partial\Omega)$, for some scalar-valued $\tilde{h} \in T^{1,q'}_{w'}(\partial\Omega)$, and where ϕ_h is given by Lemma 6.3.5. Moreover, this tangential trace is independent of the choice of the extension ϕ_h and coincides with the tangential component of the usual restriction if $u \in C^\infty(\overline{\Omega})$.

Proof. Assume that γ is defined by (6.3.3). Let $m \in T^{1,q'}_w(\partial\Omega)$. The function m can be decomposed into its normal and tangential components, i.e.,

$$m = (N\cdot m)N + h \quad \text{with} \quad N\cdot h = 0$$

with $\|h\|_{T^{1,q'}_{w'}(\partial\Omega)} \le c\|m\|_{T^{1,q'}_{w'}(\partial\Omega)}$. Then one obtains

$$\begin{aligned} |\langle \gamma(u), m \rangle_{\partial\Omega}| &= |\langle \gamma(u), h \rangle_{\partial\Omega}| \\ &= |\langle u, \Delta\phi_h \rangle - \langle \Delta_\sigma u, \phi_h \rangle| \\ &\le \|u\|_{q,w}\|\phi_h\|_{2,q',w'} + \|\Delta_\sigma u\|_{(W^{1,r'}_{\tilde{w}',0,\sigma})'}\|\phi_h\|_{1,r',\tilde{w}'} \\ &\le c\|u\|_{\tilde{W}^{q,r}_{w,\tilde{w}}}\|m\|_{T^{1,q'}_{w'}(\partial\Omega)}. \end{aligned}$$

Thus γ is continuous.

By Gauss' Theorem we know that for $u \in C^\infty(\overline{\Omega})$, $h \in T^{1,q'}_{w'}(\partial\Omega)$ and ϕ_h defined as above

$$\begin{aligned} \langle \gamma(u), h \rangle_{\partial\Omega} =& \langle \gamma(u), N\cdot\nabla\phi_h \rangle_{\partial\Omega} = \langle u, \Delta\phi_h \rangle - \langle \Delta u, \phi_h \rangle \\ =& \langle u|_{\partial\Omega}, N\cdot\nabla\phi_h \rangle_{\partial\Omega} = \langle u|_{\partial\Omega}, h \rangle_{\partial\Omega}. \end{aligned}$$

Thus the tangential component of $\gamma(u)$ is equal to the tangential component of $u|_{\partial\Omega}$ which is in particular independent of the extension of h. Since by Lemma 6.3.4 the space $C^\infty(\overline{\Omega})$ is dense in $\tilde{W}^{q,r}_{w,\tilde{w}}$ the same is true for every $u \in \tilde{W}^{q,r}_{w,\tilde{w}}$. □

The definition of normal traces is easier. If

$$u \in E^{q,r}_{w,\tilde{w}} := \{ v \in L^q_w(\Omega) \mid \operatorname{div} v \in L^r_{\tilde{w}}(\Omega) \}$$

then we can define the normal trace $u \mapsto N\cdot u|_{\partial\Omega}$ using Green's formula by

$$\langle N\cdot u|_{\partial\Omega}, v \rangle_{\partial\Omega} := \langle u|_{\partial\Omega}, Nv \rangle_{\partial\Omega} := \langle \operatorname{div} u, v \rangle + \langle u, \nabla v \rangle \quad \text{for all} \quad v \in W^{1,q'}_{w'}(\Omega). \tag{6.3.4}$$

This is well-defined since by Theorem 5.2.2 for every $\zeta \in T^{1,q'}_{w'}(\partial\Omega)$ there exists $v \in W^{1,q}_{w}(\Omega)$ with

$$v|_{\partial\Omega} = \zeta \quad \text{and} \quad \|v\|_{1,q',w'} \le c\|\zeta\|_{T^{1,q'}_{w'}}. \tag{6.3.5}$$

Moreover, since $W^{1,q}_{w}(\Omega) \hookrightarrow W^{1,r}(\Omega)$ for an appropriate $r \in (1,\infty)$, we obtain from the corresponding result in the unweighted case [47] that the right hand side in (6.3.4) is independent of the extension v.

Then it follows from (6.3.5) that the mapping

$$u \mapsto N \cdot u|_{\partial\Omega} \quad : E^{q,r}_{w,\tilde{w}} \to T^{0,q}_{w}(\partial\Omega)$$

is continuous. Using the above theorem for $u \in \tilde{W}^{q,r}_{w,\tilde{w}} \cap E^{q,r}_{w,\tilde{w}}$ we write $u|_{\partial\Omega} = g$ if

$$\langle \gamma(u), h\rangle_{\partial\Omega} = \langle g, h\rangle_{\partial\Omega} \ \text{ for all } \ h \in T^{1,q'}_{w'}(\partial\Omega) \ \text{ with } \ h\cdot N = 0 \ \text{ and } \ u \cdot N|_{\partial\Omega} = g \cdot N. \tag{6.3.6}$$

With this notation we also define the operator

$$\mathrm{tr}\ : \tilde{W}^{q,r}_{w,\tilde{w}} \cap E^{q,r}_{w,\tilde{w}} \to T^{0,q}_{w}(\Omega), \quad u \mapsto g.$$

Proposition 6.3.7. *Let u be a very weak solution to the Stokes problem corresponding to the data $\langle f,\phi\rangle = \langle F,\phi\rangle - \langle g, N\cdot\nabla\phi\rangle_{\partial\Omega}$ and $\langle k,\psi\rangle = \langle K,\psi\rangle - \langle g, N\cdot\psi\rangle_{\partial\Omega}$ with $F \in W^{-1,r}_{\tilde{w}}(\Omega)$, $K \in L^{r}_{\tilde{w}}(\Omega)$, $g \in T^{0,q}_{w}(\partial\Omega)$.*

Then $u \in \tilde{W}^{q,r}_{w,\tilde{w}} \cap E^{q,r}_{w,\tilde{w}}$ and $u|_{\partial\Omega} = g$.

Proof. By definition, u is the solution to the variational problem

$$\begin{aligned} -\langle u,\Delta\phi\rangle &= \langle F,\phi\rangle - \langle g, N\cdot\nabla\phi\rangle_{\partial\Omega}, \quad \text{for all } \phi \in Y^{2,q'}_{w',\sigma}(\Omega) \ \text{ and} \\ -\langle u,\nabla\psi\rangle &= \langle K,\psi\rangle - \langle g, N\cdot\psi\rangle_{\partial\Omega}, \qquad \text{for all } \psi \in W^{1,q'}_{w'}(\Omega). \end{aligned}$$

Inserting $\phi \in C^{\infty}_{0,\sigma}(\Omega)$ into the first equation we obtain that $[\phi \mapsto \langle\Delta u,\phi\rangle = -\langle F,\phi\rangle]$ is extendable to an element of $(W^{1,r'}_{\tilde{w},0,\sigma}(\Omega))'$. Thus $u \in \tilde{W}^{q,r}_{w,\tilde{w}}$ and by the definition of the tangential trace we have

$$\langle\gamma(u), N\cdot\nabla\phi\rangle_{\partial\Omega} = \langle u,\Delta\phi\rangle - \langle\Delta_\sigma u,\phi\rangle = \langle u,\Delta\phi\rangle + \langle F,\phi\rangle = \langle g, N\cdot\nabla\phi\rangle_{\partial\Omega}$$

for all $\phi \in Y^{2,q'}_{w',\sigma}(\Omega)$. Using the second equation one shows that $N\cdot u|_{\partial\Omega} = N\cdot g$. □

Remark 6.3.8. 1. It is not difficult to see that the space $\tilde{W}^{q,r}_{w,\tilde{w}}$ is equal to the space of very weak solutions to the Stokes problem with respect to data

$$f = [\phi \mapsto \langle F,\phi\rangle - \langle g, N\cdot\nabla\phi\rangle_{\partial\Omega}]$$

with $F \in W^{-1,r}_{\tilde{w}}(\Omega)$ and $g \in T^{0,q}_{w}(\partial\Omega)$ and $k \in W^{-1,q}_{w,0}(\Omega)$. Indeed, let $u \in \tilde{W}^{q,r}_{w,\tilde{w}}$ and let $F \in W^{-1,r}_{\tilde{w}}(\Omega)$ be an extension of $-\Delta u|_{C^{\infty}_{0,\sigma}(\Omega)}$. Then setting $g := \gamma u \in T^{0,q}_{w}(\Omega)$ we obtain by the definition of γ

$$-\langle u,\Delta\phi\rangle = \langle F,\phi\rangle - \langle g, N\cdot\nabla\phi\rangle_{\partial\Omega} \ \text{ for every } \ \phi \in Y^{2,q'}_{w',\sigma}(\Omega).$$

2. In [30] the unweighted case is trated. There the space in which the traces are well-defined is defined in a different way. We repeat this definition and show that the out-coming space is the same in the case $w = \tilde{w} = 1$.

For $u \in W^{1,q}(\Omega)$ one sets

$$\|\mathcal{A}_r^{-\frac{1}{2}} P_r \Delta u\|_{L^r_\sigma(\Omega)} = \sup_{0 \neq v \in L^{r'}_\sigma} \left(\frac{\langle \nabla u, \nabla \mathcal{A}_{r'}^{-\frac{1}{2}} v \rangle}{\|v\|_{L^{r'}_\sigma(\Omega)}} \right),$$

where $\mathcal{A}_r$ stands for the Stokes operator and P_r for the Helmholtz projection in $L^r(\Omega)$ and $\frac{1}{r} \leq \frac{1}{n} + \frac{1}{q}$. Note that r is chosen such that by the Sobolev embedding theorems [1] one has

$$W^{1,r}(\Omega) \hookrightarrow L^q(\Omega).$$

Then one defines

$$\widehat{W}^{1,q}(\Omega) := \overline{W^{1,q}(\Omega)}^{\|\cdot\|_{\widehat{W}^{1,q}(\Omega)}} \quad \text{where} \quad \|u\|_{\widehat{W}^{1,q}(\Omega)} := \|u\|_q + \|\mathcal{A}_r^{-\frac{1}{2}} P_r \Delta u\|_r.$$

For $u \in C^\infty(\overline{\Omega})$ one has

$$\begin{aligned} \|\Delta u|_{C^\infty_{0,\sigma}}\|_{(W^{1,r'}_{0,\sigma})'} &= \sup_{\phi \in C^\infty_{0,\sigma}, \|\phi\|_{1,r'}=1} |\langle \Delta u, \phi \rangle| \\ &\sim \sup_{\psi \in C^\infty_{0,\sigma}, \|\psi\|_{r'}=1} |\langle P_r \Delta u, \mathcal{A}_{r'}^{-\frac{1}{2}} \psi \rangle| = \|\mathcal{A}_r^{-\frac{1}{2}} P_r \Delta u\|_r, \end{aligned}$$

where we have used that by [33] one has $\|\mathcal{A}_{r'}^{\frac{1}{2}} \cdot \|_{r'} \sim \|\nabla \cdot \|_{r'}$.

Thus in the unweighted case these norms are equivalent and by the density shown in Lemma 6.3.4 the spaces are equal.

7 Weighted Bessel Potential Spaces

7.1 Weighted Bessel Potential Spaces

For $\xi \in \mathbb{R}^n$ we set $\langle \xi \rangle := (1 + |\xi|^2)^{\frac{1}{2}}$. On the space $\mathcal{S}'(\mathbb{R}^n; \mathbb{R})$ of temperate distributions we define for all $\beta \in \mathbb{R}$ the operator

$$\Lambda^\beta f = \mathcal{F}^{-1} \langle \xi \rangle^\beta \mathcal{F} f, \quad f \in S'(\mathbb{R}^n; \mathbb{R}),$$

where $\mathcal{F}$ stands for the Fourier transformation on $\mathcal{S}'(\mathbb{R}^n; \mathbb{R})$. Then for $1 < q < \infty$, $w \in A_q$ and $\beta \in \mathbb{R}$ the weighted Bessel potential space is given by

$$H_w^{\beta,q}(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n; \mathbb{R}) \mid \|f\|_{H_w^{\beta,q}(\mathbb{R}^n)} := \|\Lambda^\beta f\|_{q,w,\mathbb{R}^n} < \infty \right\}.$$

Theorem 7.1.1. *If $1 < q < \infty$, $w \in A_q$, $l, k \in \mathbb{Z}$ and $l < \beta < k$ then*

$$\left[H_w^{l,q}(\mathbb{R}^n), H_w^{k,q}(\mathbb{R}^n) \right]_\theta = H_w^{\beta,q}(\mathbb{R}^n),$$

where $\theta = \frac{\beta - l}{k - l}$. The norms are equivalent with A_q-consistent equivalence constants.

Proof. This can be proven analogously to [50, Proposition 13.6.2]. For the weighted version in the case $l = 0$ and $k \in \mathbb{N}$ see also [25, Satz 8.3]. The proof given there can be repeated to obtain the more general assertion of this theorem. It is based on the boundedness of the purely imaginary powers Λ^{iy} in $L_w^q(\mathbb{R}^n)$ which is a consequence of the weighted Multiplier Theorem 3.2.4. Thus rereading the proof one also obtains the A_q-consistence of the constants. $\square$

For an extension domain Ω we define the weighted Bessel potential space on Ω by

$$H_w^{\beta,q}(\Omega) = \left\{ g|_\Omega \mid g \in H_w^{\beta,q}(\mathbb{R}^n) \right\}$$

equipped with the norm

$$\|u\|_{H_w^{\beta,q}(\Omega)} := \inf \left\{ \|U\|_{H_w^{\beta,q}(\mathbb{R}^n)} \mid U \in H_w^{\beta,q}(\mathbb{R}^n),\ U|_\Omega = u \right\}.$$

Note that if $\beta < 0$ then the restriction $g|_\Omega$ has to be understood in the sense of distributions as $g|_{C_0^\infty(\Omega)}$.

Moreover, we set

$$H_{w,0}^{\beta,q}(\Omega) = \overline{(C_0^\infty(\Omega))}^{H_w^{\beta,q}(\mathbb{R}^n)}, \quad \beta \in \mathbb{R},$$

equipped with the norm $\|\cdot\|_{\beta,q,w,0,\Omega} := \|E_0(\cdot)\|_{\beta,q,w,\mathbb{R}^n}$, where E_0 denotes the extension of a function by 0 to the whole space $\mathbb{R}^n$. The space $H_{w,0}^{\beta,q}(\Omega)$ is a reflexive Banach space being a closed subspace of $H_w^{\beta,q}(\mathbb{R}^n)$, which is reflexive since it is isomorphic to $L_w^q(\Omega)$.

Note that by (7.4.2) below this norm is in general not equivalent to $\|\cdot\|_{\beta,q,w,\Omega}$. Moreover, if $\beta < 0$ the space $H^{\beta,q}_{w,0}(\Omega)$ does not consist of distributions on Ω but of distributions on $\mathbb{R}^n$ supported by $\overline{\Omega}$.

We choose this definition because in this way one obtains a good behavior of the dual spaces and interpolation properties, see Lemma 7.3.2 below.

Theorem 7.1.2. *Let Ω be an extension domain, $1 < q < \infty$, $w \in A_q$.*

1. *For $k \in \mathbb{N}_0$ one has $H^{k,q}_w(\Omega) = W^{k,q}_w(\Omega)$ and $H^{k,q}_{w,0}(\Omega) = W^{k,q}_{w,0}(\Omega)$ with equivalent norms.*
2. *For $k \in \mathbb{N}$, $0 < \beta < k$ one has*

$$H^{\beta,q}_w(\Omega) = [L^q_w(\Omega), W^{k,q}_w(\Omega)]_{\frac{\beta}{k}}.$$

3. *The spaces $H^{\beta,q}_w(\Omega)$, $\beta > 0$, are independent of the values of the weight function $w \in A_q$ outside Ω, i.e., if $w_1, w_2 \in A_q$, $w_1|_\Omega = w_2|_\Omega$ then $H^{\beta,q}_{w_1}(\Omega) = H^{\beta,q}_{w_2}(\Omega)$ with equivalent norms.*

Proof. All assertions can be found in [25, 8.2.2] except for the assertion on $H^{k,q}_{w,0}(\Omega)$ in 1. However, since one has $H^{k,q}_w(\mathbb{R}^n) = W^{k,q}_w(\mathbb{R}^n)$ with equivalent norms, the equation $H^{k,q}_{w,0}(\Omega) = W^{k,q}_{w,0}(\Omega)$ follows from the definition of $H^{k,q}_{w,0}(\Omega)$. ☐

Corollary 7.1.3. *Let $\beta \in [0,1]$. Then*

$$\Lambda^\beta : [L^q_w(\mathbb{R}^n), W^{1,q}_w(\mathbb{R}^n)]_\beta \to L^q_w(\mathbb{R}^n)$$

is continuous with A_q-consistent continuity constant.

Proof. As stated in Theorems 7.1.2.1 one has $W^{1,q}_w(\mathbb{R}^n) = H^{1,q}_w(\mathbb{R}^n)$ with equivalent constants. However, in the whole space case this is proved by the use of the Multiplier Theorem. Thus re-reading this proof shows that the equivalence constants are A_q-consistent.

By Theorem 7.1.1 one has

$$\|\Lambda^\beta u\|_{L^q_w(\mathbb{R}^n)} = \|u\|_{H^{\beta,q}_w(\mathbb{R}^n)} \le c\|u\|_{[L^q_w(\mathbb{R}^n), W^{1,q}_w(\mathbb{R}^n)]_\beta},$$

where $c > 0$ is A_q-consistent. ☐

7.2 Some Technical Lemmas

In Section 9.4 the proof of higher regularity with inhomogeneous boundary values requires some interpolation and continuity results which are provided in this section. In particular, we concentrate on the A_q-consistence of equivalence and continuity constants. Thus we now present a collection of some technical lemmas in this context.

Throughout Section 7.2 let $1 < q < \infty$ and $w \in A_q$.

Lemma 7.2.1. *Let Ω and $\mathcal{O}$ equal to $\mathbb{R}^n$, $\mathbb{R}^n_+$ a bent half space H_σ or a bounded $C^{1,1}$-domain and let $\psi : \Omega \to \mathcal{O}$ be a $C^{0,1}$-diffeomorphism. Then*

$$C_\psi : u \mapsto u \circ \psi : \quad [L^q_w(\mathcal{O}), W^{1,q}_w(\mathcal{O})]_\beta \to [L^q_{w\circ\psi}(\Omega), W^{1,q}_{w\circ\psi}(\Omega)]_\beta \tag{7.2.1}$$

is continuous for every $\beta \in [0,1]$ with a constant independent of the weight function w.

Proof. The assertion follows from Lemma 3.3.6 if $\beta = 0$ and $\beta = 1$ and from interpolation if $\beta \in (0,1)$. □

Lemma 7.2.2. *Let $\phi \in C^\infty_0(\mathbb{R}^n)$, Ω a bounded $C^{1,1}$-domain and H_α a bent half space such that*

$$(\partial H_\alpha) \cap (\operatorname{supp} \phi) = (\partial\Omega) \cap (\operatorname{supp} \phi).$$

Let $M_\phi : S'(\mathbb{R}^n;\mathbb{R}) \to S'(\mathbb{R}^n;\mathbb{R})$, $u \mapsto \phi u$, be the multiplication operator. Moreover, let $l, k \in \mathbb{N}_0$, $l < k$.

1. *The operator*

$$M_\phi : [W^{l,q}_w(\Omega), W^{k,q}_w(\Omega)]_\theta \to [W^{l,q}_w(H_\alpha), W^{k,q}_w(H_\alpha)]_\theta$$

   is continuous with continuity constant independent of the weight function w.

2. *For $w_1, w_2 \in A_q$ with $w_1 = w_2$ on $\operatorname{supp} \phi$ the operator*

$$M_\phi : [W^{l,q}_{w_1}(\Omega), W^{k,q}_{w_1}(\Omega)]_\theta \to [W^{l,q}_{w_2}(\Omega), W^{k,q}_{w_2}(\Omega)]_\theta$$

   is continuous with a continuity constant that is independent of w_1 and w_2.

Proof. The assertions are clear for $\theta = 0$ and $\theta = 1$, the rest follows by interpolation. □

Lemma 7.2.3. *Let $\beta \in [0,1]$. Then the even extension,*

$$E_e : [L^q_w(\mathbb{R}^n_+), W^{1,q}_w(\mathbb{R}^n_+)]_\beta \to [L^q_{w^*}(\mathbb{R}^n), W^{1,q}_{w^*}(\mathbb{R}^n)]_\beta$$
$$E_e u(x) = \begin{cases} u(x) & \text{on } \mathbb{R}^n_+ \\ u(x', -x_n) & \text{on } \mathbb{R}^n_-, \end{cases}$$

is continuous and the continuity constant is independent of w; for the definition of w^* see (3.1.3).

Proof. This follows from interpolation and the assertions for $\beta = 0$ and $\beta = 1$. The continuity constant is 2. □

Lemma 7.2.4. *Let $\Omega = \mathbb{R}^n$ or a bounded $C^{1,1}$-domain. Then the norm in $W^{1,q}_w(\Omega)$ is equivalent to the one in $[L^q_w(\Omega), W^{2,q}_w(\Omega)]_{\frac{1}{2}}$ with an equivalence constant depending A_q-consistently on w.*

Proof. We begin with the case $\Omega = \mathbb{R}^n$. In this case by Theorem 7.1.1 one has

$$[L^q_w(\mathbb{R}^n), H^{k,q}_w(\mathbb{R}^n)]_{\frac{\beta}{k}} = H^{\beta,q}_w(\mathbb{R}^n)$$

with equivalence constants of the norms that depend A_q-consistently on the weight functions. Moreover, if one repeats the arguments in the proof of [25, Lemma 8.1], one obtains that the norms in $H^{k,q}_w(\mathbb{R}^n)$ and in $W^{k,q}_w(\mathbb{R}^n)$ are equivalent with A_q-consistent constants, since the proof of this is based on the Multiplier Theorem.

For the case of a bounded domain Ω we take an open covering $(U_j)_{j=1}^m$ of $\overline{\Omega}$, a collection of charts $(\alpha_j)_{j=1}^m$, $\alpha_j : V_j \to U_j$, and a partition of unity $(\phi_j)_{j=1}^m$ subordinate to the covering $(U_j)_j$. Assume that each α_j is extended to a $C^{1,1}$-diffeomorphism on $\mathbb{R}^n$. Moreover, let

$$E_{\mathbb{R}^n_+,j} : W^{2,q}_{w\circ\alpha_j}(\mathbb{R}^n_+) \to W^{2,q}_{\widetilde{w\circ\alpha_j}}(\mathbb{R}^n)$$

be the continuous extension operator defined in Lemma 3.3.7 where $\widetilde{w\circ\alpha_j}$ is the weight function constructed in Lemma 3.3.7 starting with $w\circ\alpha_j$. In particular, this extension operator extends to a continuous operator

$$E_{\mathbb{R}^n_+,j} : L^q_{w\circ\alpha_j}(\mathbb{R}^n_+) \to L^q_{\widetilde{w\circ\alpha_j}}(\mathbb{R}^n).$$

We define the mapping

$$\begin{aligned} P : \prod_{j=1}^m W^{2,q}_{\widetilde{w\circ\alpha_j}}(\mathbb{R}^n) &\to W^{2,q}_w(\Omega), \\ (u_1, ..., u_m) &\mapsto \sum_{j=1}^m \psi_j R_\Omega(u_j \circ \alpha_j^{-1}), \end{aligned}$$

where $\psi_j \in C_0^\infty(U_j)$ with $\psi_j \equiv 1$ on $\operatorname{supp}\phi_j$ and R_Ω denotes the restriction of functions defined on $\mathbb{R}^n$ to Ω. The operator P is meant to take the role of the retraction as in Theorem 2.3.1.5. Note that $\widetilde{w\circ\alpha_j}\circ\alpha_j^{-1} = w$ on $U_j \cap \Omega \supset \operatorname{supp}\psi_j$. Then a coretraction is given by

$$\begin{aligned} I : W^{2,q}_w(\Omega) &\to \prod_{j=1}^m W^{2,q}_{\widetilde{w\circ\alpha_j}}(\mathbb{R}^n) \\ u &\mapsto \Big(E_{\mathbb{R}^n_+,1}((\phi_1 u)\circ\alpha_1), ..., E_{\mathbb{R}^n_+,m}((\phi_m u)\circ\alpha_m)\Big). \end{aligned}$$

Then P and I are continuous by the Lemmas 3.3.7, 7.2.1 and 7.2.2 with A_q-consistent continuity constants also if they are considered as operators

$$P : \prod_{j=1}^m L^q_{\widetilde{w\circ\alpha_j}}(\mathbb{R}^n) \to L^q_w(\Omega) \quad \text{and} \quad I : L^q_w(\Omega) \to \prod_{j=1}^m L^q_{\widetilde{w\circ\alpha_j}}(\mathbb{R}^n).$$

Moreover, for $u \in L^q_w(\Omega)$ one has

$$PIu = \sum_{j=1}^m \psi_j R_\Omega(E_{\mathbb{R}^n_+,j}((\phi_j u)\circ\alpha_j)\circ\alpha_j^{-1}) = \sum_{j=1}^m \psi_j\phi_j u = u.$$

Thus, by Theorem 2.3.1.5. and using the assertion for $\Omega = \mathbb{R}^n$ we find

$$\begin{aligned}
[L^q_w(\Omega), W^{2,q}_w(\Omega)]_{\frac{1}{2}} &= P\left[\prod_{j=1}^m L^q_{\widetilde{w\circ\alpha_j}}(\mathbb{R}^n), \prod_{j=1}^m W^{2,q}_{\widetilde{w\circ\alpha_j}}(\mathbb{R}^n)\right]_{\frac{1}{2}} \\
&= P\left(\prod_{j=1}^m W^{1,q}_{\widetilde{w\circ\alpha_j}}(\Omega)\right) = W^{1,q}_w(\Omega),
\end{aligned}$$

where the last equation is clear by the definition of P. The constants are A_q consistent since so are the constants of P and I. □

Lemma 7.2.5. *Let $\Omega = \mathbb{R}^n$ or a bounded $C^{1,1}$-domain and let $\beta \in [1,2]$. Then for every $u \in H^{\beta,q}_w(\Omega)$ one has the estimate*

$$\|u\|_{H^{\beta,q}_w(\Omega)} \le c\left(\|u\|_{H^{\beta-1,q}_w(\Omega)} + \|\nabla u\|_{H^{\beta-1,q}_w(\Omega)}\right),$$

where $c = c(\beta, q, w, \Omega)$.

Proof. We begin to show the inequality in $\mathbb{R}^n$. Let $u \in H^{\beta,q}_w(\mathbb{R}^n)$. Then one has by the Multiplier theorem 3.2.4

$$\begin{aligned}
\|u\|_{H^{\beta,q}_w(\mathbb{R}^n)} &= \|\Lambda\Lambda^{\beta-1}u\|_{L^q_w(\mathbb{R}^n)} \\
&= \left\|\mathcal{F}^{-1}\left(\frac{1}{\sqrt{1+|\xi|^2}} + \sum_{j=1}^n \frac{\xi_j}{\sqrt{1+|\xi|^2}}\xi_j\right)\mathcal{F}\Lambda^{\beta-1}u\right\|_{L^q_w(\mathbb{R}^n)} \\
&\le c\left(\|\Lambda^{\beta-1}u\|_{L^q_w(\mathbb{R}^n)} + \sum_{j=1}^n \|\mathcal{F}\xi_j\mathcal{F}\Lambda^{\beta-1}u\|_{L^q_w(\mathbb{R}^n)}\right) \\
&\le c\left(\|u\|_{H^{\beta-1,q}_w(\mathbb{R}^n)} + \|\nabla u\|_{H^{\beta-1,q}_w(\mathbb{R}^n)}\right).
\end{aligned} \tag{7.2.2}$$

This is the assertion for $\mathbb{R}^n$.

In Lemma 3.3.7 it has been shown that the extension operator $E_{\mathbb{R}^n_+}$ defined by

$$E_{\mathbb{R}^n_+}u(x) = \begin{cases} u(x) & \text{for } \; x_n > 0 \\ \sum_{j=1}^3 \lambda_j u(x', -jx_n) & \text{for } \; x_n < 0, \end{cases}$$

where λ_j, $j = 1, ..., 3$, is chosen such that $\sum_{j=1}^3 \lambda_j(-j)^l = 1$ for $l = 0, ..., 3$, is continuous as an operator

$$E_{\mathbb{R}^n_+} : W^{k,q}_w(\mathbb{R}^n_+) \to W^{k,q}_{\tilde{w}}(\mathbb{R}^n), \qquad k = 0, 1, 2,$$

where $\tilde{w}_{\mathbb{R}^n_+}$ is given by

$$\tilde{w} = \begin{cases} w(x', x_n) & \text{if } \; x_n > 0 \\ \min_{j=1,\dots,3} w(x', -jx_n) & \text{if } \; x_n < 0. \end{cases}$$

The continuity constant of $E_{\mathbb{R}^n_+}$ and $A_q(\tilde{w}_{\mathbb{R}^n_+})$ depend A_q-consistently on w.

Analogously, one shows that the extension operator

$$\tilde{E}_{\mathbb{R}^n_+} : v(x) = (v', v_n)(x', x_n) \mapsto \begin{cases} v(x', x_n) & \text{on } \mathbb{R}^n_+ \\ \begin{pmatrix} E(v')(x', x_n) \\ \sum_{j=1}^{3} \lambda_j (-j) v_n(x', -jx_n) \end{pmatrix} & \text{on } \mathbb{R}^n_- \end{cases}$$

is continuous as an operator

$$\tilde{E}_{\mathbb{R}^n_+} : W^{k,q}_w(\mathbb{R}^n_+) \to W^{k,q}_{\tilde{w}}(\mathbb{R}^n), \qquad k = 0, 1.$$

Interpolation shows that

$$\tilde{E}_{\mathbb{R}^n_+} : H^{\beta-1,q}_{\tilde{w}}(\mathbb{R}^n_+) \to H^{\beta-1,q}_{\tilde{w}}(\mathbb{R}^n),$$

and by construction one has $\nabla E_{\mathbb{R}^n_+} = \tilde{E}_{\mathbb{R}^n_+} \nabla$.

To prove the result for a bounded domain Ω let $(\alpha_j)_{j=1}^m$ be a collection of charts and $(\psi_j)_{j=1}^m$ a decomposition of unity subordinate to the corresponding covering of Ω.

Thus we can calculate using Theorem 7.1.2.1 and 2 with reiteration together with the Lemmas 7.2.1 and 7.2.2

$$\begin{aligned} \|u\|_{[W^{1,q}_w(\Omega), W^{2,q}_w(\Omega)]_{\beta-1}} &\le \sum_{j=1}^m \|\psi_j u\|_{[W^{1,q}_w(\Omega), W^{2,q}_w(\Omega)]_{\beta-1}} \\ &\le c \sum_{j=1}^m \|\psi_j u\|_{H^{\beta,q}_w(\Omega)} \le c \sum_{j=1}^m \|\psi_j u\|_{H^{\beta,q}_w(H_{\alpha_j})} \\ &\le c \sum_{j=1}^m \|(\psi_j u) \circ \alpha_j\|_{H^{\beta,q}_{w\circ\alpha_j}(\mathbb{R}^n_+)} \le c \sum_{j=1}^m \|E_{\mathbb{R}^n_+}((\psi_j u) \circ \alpha_j)\|_{H^{\beta,q}_{\widetilde{w\circ\alpha_j}}(\mathbb{R}^n)}. \end{aligned}$$

Using the result in the whole space case and $\widetilde{w \circ \alpha_j} = w$ on $\operatorname{supp} \phi_j$ we obtain

$$\begin{aligned} \|u\|_{H^{\beta,q}_w(\Omega)} &\le c\|u\|_{[W^{1,q}_w(\Omega), W^{2,q}_w(\Omega)]_{\beta-1}} \\ &\le c \sum_{j=1}^m \Big(\|E_{\mathbb{R}^n_+}((\psi_j u) \circ \alpha_j)\|_{H^{\beta-1,q}_{\widetilde{w\circ\alpha_j}}(\mathbb{R}^n)} + \|\nabla E_{\mathbb{R}^n_+}((\psi_j u) \circ \alpha_j)\|_{H^{\beta-1,q}_{\widetilde{w\circ\alpha_j}}(\mathbb{R}^n)} \Big) \\ &\le c \sum_{j=1}^m \Big(\|E_{\mathbb{R}^n_+}((\psi_j u) \circ \alpha_j)\|_{H^{\beta-1,q}_{\widetilde{w\circ\alpha_j}}(\mathbb{R}^n)} + \|\tilde{E}_{\mathbb{R}^n_+} \nabla((\psi_j u) \circ \alpha_j)\|_{H^{\beta-1,q}_{\widetilde{w\circ\alpha_j}}(\mathbb{R}^n)} \Big) \\ &\le c \sum_{j=1}^m \Big(\|(\psi_j u)\|_{H^{\beta-1,q}_{\widetilde{w\circ\alpha_j}\circ\alpha_j^{-1}}(H_{\alpha_j})} + \|\nabla(\psi_j u)\|_{H^{\beta-1,q}_{\widetilde{w\circ\alpha_j}\circ\alpha_j^{-1}}(H_{\alpha_j})} \Big) \\ &\le c \sum_{j=1}^m \Big(\|u\|_{H^{\beta-1,q}_w(\Omega)} + \|\nabla\psi_j \cdot u\|_{H^{\beta-1,q}_w(\Omega)} + \|\psi_j \cdot \nabla u\|_{H^{\beta-1,q}_w(\Omega)} \Big) \\ &\le c \sum_{j=1}^m \Big(\|u\|_{H^{\beta-1,q}_w(\Omega)} + \|u\|_{H^{\beta-1,q}_w(\Omega)} + \|\nabla u\|_{H^{\beta-1,q}_w(\Omega)} \Big) \\ &\le c(\|u\|_{H^{\beta-1,q}_w(\Omega)} + \|\nabla u\|_{H^{\beta-1,q}_w(\Omega)}). \end{aligned}$$

This is the asserted estimate. $\square$

7.3 Bessel Potential Spaces of Negative Order

Throughout this section let $1 < q < \infty$ and $w \in A_q$.

Lemma 7.3.1. *For $\beta > 0$ one has $H_w^{-\beta,q}(\mathbb{R}^n) = \left(H_{w'}^{\beta,q'}(\mathbb{R}^n)\right)'$ isometrically.*

Proof. For $u \in H_w^{-\beta,q}(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n;\mathbb{R})$ and $\phi \in \mathcal{S}(\mathbb{R}^n;\mathbb{R})$ we have

$$|\langle u, \phi\rangle| = |\langle u, \Lambda^{-\beta}\Lambda^{\beta}\phi\rangle| = |\langle \Lambda^{-\beta}u, \Lambda^{\beta}\phi\rangle| \leq \|\Lambda^{-\beta}u\|_{q,w}\|\Lambda^{\beta}\phi\|_{q',w'}.$$

Thus $u \in \left(H_{w'}^{\beta,q'}(\mathbb{R}^n)\right)'$ and $\|u\|_{(H_{w'}^{\beta,q'}(\mathbb{R}^n))'} \leq \|u\|_{-\beta,q,w}$.

Vice versa, if $u \in \left(H_{w'}^{\beta,q'}(\mathbb{R}^n)\right)' \subset \mathcal{S}'(\mathbb{R}^n;\mathbb{R})$ then one has for every $\phi \in \mathcal{S}(\mathbb{R}^n;\mathbb{R})$

$$|\langle \Lambda^{-\beta}u, \phi\rangle| = |\langle u, \Lambda^{-\beta}\phi\rangle| \leq \|u\|_{(H_{w'}^{\beta,q'}(\mathbb{R}^n))'}\|\Lambda^{-\beta}\phi\|_{\beta,q',w'} = \|u\|_{(H_{w'}^{\beta,q'}(\mathbb{R}^n))'}\|\phi\|_{q',w'}.$$

One obtains $\Lambda^{-\beta}u \in L_w^q(\mathbb{R}^n)$ and consequently $u \in H_w^{-\beta,q}(\mathbb{R}^n)$. $\square$

Recall that for $\beta > 0$ the weighted Bessel potential space of negative order on an extension domain Ω is defined by

$$H_w^{-\beta,q}(\Omega) = \left\{ u|_{C_0^\infty(\Omega)} \mid u \in H_w^{-\beta,q}(\mathbb{R}^n) \right\},$$

equipped with the norm

$$\|u\|_{-\beta,q,w,\Omega} = \inf\left\{ \|v\|_{-\beta,q,w,\mathbb{R}^n} \mid v \in H_w^{-\beta,q}(\mathbb{R}^n),\ v|_{C_0^\infty(\Omega)} = u \right\}.$$

Lemma 7.3.2. *For $\beta \in \mathbb{R}$ one has*

$$H_w^{-\beta,q}(\Omega) = \left(H_{w',0}^{\beta,q'}(\Omega)\right)', \tag{7.3.1}$$

with equivalent norms.

Moreover, for $k \in \mathbb{N}$ one has $H_w^{-k,q}(\Omega) = W_w^{-k,q}(\Omega)$.

Proof. Let $u \in H_w^{-\beta,q}(\Omega)$. Then by definition there exists $U \in H_w^{-\beta,q}(\mathbb{R}^n)$ such that $U|_{C_0^\infty(\Omega)} = u$ with

$$\begin{aligned} 2\|u\|_{-\beta,q,w,\Omega} &\geq \|U\|_{-\beta,q,w,\mathbb{R}^n} = \sup_{\phi\in\mathcal{S}(\mathbb{R}^n),\|\phi\|_{\beta,q',w',\mathbb{R}^n}\leq 1} \langle U, \phi\rangle \\ &\geq \sup_{\phi\in C_0^\infty(\Omega),\|\phi\|_{\beta,q',w',\mathbb{R}^n}\leq 1} \langle u, \phi\rangle = \|u\|_{(H_{w',0}^{\beta,q'}(\Omega))'} \end{aligned}$$

using Lemma 7.3.1 and Hahn-Banach's Theorem. Thus $u \in (H_{w',0}^{\beta,q'}(\Omega))'$.

Vice versa, by Hahn-Banach's theorem every $u \in \left(H_{w',0}^{\beta,q'}(\Omega)\right)'$ can be extended to an element

$$U \in \left(H_{w'}^{\beta,q'}(\mathbb{R}^n)\right)' = H_w^{-\beta,q}(\mathbb{R}^n) \text{ with } \|U\|_{-\beta,q,w,\mathbb{R}^n} = \|u\|_{(H_{w',0}^{\beta,q'}(\Omega))'}.$$

Then a similar calculation as above yields $u \in H_w^{-\beta,q}(\Omega)$ with $\|u\|_{-\beta,q,w,\Omega} \leq \|u\|_{(H_{w',0}^{\beta,q'}(\Omega))'}$.

To obtain the result for $k \in \mathbb{N}$ one combines the first assertion with Theorem 7.1.2.1. $\square$

Lemma 7.3.2 also yields the completeness of $H_w^{-\beta,q}(\Omega)$ in the case $\beta > 0$.

Lemma 7.3.3. *Let Ω be a bounded $C^{1,1}$-domain or the half space. There exists a continuous linear extension operator*

$$E : H_w^{-1,q}(\Omega) \to H_w^{-1,q}(\mathbb{R}^n)$$

such that $Eu|_{C_0^\infty(\Omega)} = u$ for all $u \in H_w^{-1,q}(\Omega)$ and which is also continuous as a mapping $E : H_w^{1,q}(\Omega) \to H_w^{1,q}(\mathbb{R}^n)$.

Proof. We begin with showing the assertion for the half space $\Omega = \mathbb{R}^n_+$.

By [27] for every $f \in W_w^{-1,q}(\mathbb{R}^n_+)$ there exists a unique $u \in W_{w,0}^{1,q}(\mathbb{R}^n_+)$ solving the equation $(1-\Delta)u = f$. This solution u depends linearly on f and fulfills the estimate $\|u\|_{1,q,w} \leq c\|f\|_{-1,q,w}$. We write $u = (1-\Delta_D)^{-1}f$. By Theorem 4.1.2 $f \in W_w^{1,q}(\mathbb{R}^n_+)$ yields $u \in W_w^{3,q}(\mathbb{R}^n_+)$ with $\|u\|_{3,q,w} \leq c\|f\|_{1,q,w}$.

To construct E we remind that by [9] there exists a linear continuous extension operator

$$\tilde{E} : W_w^{1,q}(\mathbb{R}^n_+) \to W_w^{1,q}(\mathbb{R}^n) \text{ and } \tilde{E} : W_w^{3,q}(\mathbb{R}^n_+) \to W_w^{3,q}(\mathbb{R}^n) \text{ with } \tilde{E}u|_{\mathbb{R}^n_+} = u.$$

Now we set

$$Eu = (1-\Delta)\tilde{E}(1-\Delta_D)^{-1}u, \quad u \in H_w^{-1,q}(\mathbb{R}^n_+).$$

For $\phi \in C_0^\infty(\mathbb{R}^n_+)$ one has

$$\begin{aligned}
\langle Eu, \phi\rangle_{\mathbb{R}^n} &= \langle (1-\Delta)\tilde{E}(1-\Delta_D)^{-1}u, \phi\rangle_{\mathbb{R}^n} \\
&= \langle \tilde{E}(1-\Delta_D)^{-1}u, (1-\Delta)\phi\rangle_{\mathbb{R}^n} \\
&= \langle (1-\Delta_D)^{-1}u, (1-\Delta)\phi\rangle_{\mathbb{R}^n} \\
&= \langle u, \phi\rangle_{\mathbb{R}^n}.
\end{aligned}$$

Thus E has the asserted properties on the half space $\mathbb{R}^n_+$.

For a bounded $C^{1,1}$-domain Ω we take a collection of charts $(\alpha_j)_{j=1}^m$ and a decomposition of unity $(\psi_j)_{j=1}^m$ subordinate to the corresponding covering of $\overline{\Omega}$. Then for $u \in W_w^{1,q}(\Omega)$ we set

$$E_\Omega u = \sum_{j=1}^m E_{\mathbb{R}^n_+}((u\psi_j)\circ\alpha_j)\circ\alpha_j^{-1},$$

where $E_{\mathbb{R}^n_+} : W_{w\circ\alpha_j}^{1,q}(\mathbb{R}^n_+) \to W_{w\circ\alpha_j}^{1,q}(\mathbb{R}^n)$ is the operator just constructed. Obviously $E_\Omega : W_w^{1,q}(\Omega) \to W_w^{1,q}(\mathbb{R}^n)$ is continuous. Moreover, it follows from Lemma 3.3.6 that $u \mapsto u\circ\alpha_j$ is a continuous operation from $W_w^{-1,q}(\Omega) \to W_{w\circ\alpha_j}^{-1,q}(\alpha_j^{-1}(\Omega))$. This shows the continuity of $E_\Omega : W_w^{-1,q}(\Omega) \to W_w^{-1,q}(\mathbb{R}^n)$, and combined with Lemma 7.3.2 the proof is complete. □

Theorem 7.3.4. *Let $1 < q < \infty$, $w \in A_q$, $-1 \leq \beta \leq 1$ and $\Omega = \mathbb{R}^n_+$ or a bounded $C^{1,1}$-domain. Then*

1. $[H_w^{-1,q}(\Omega), H_w^{1,q}(\Omega)]_\theta = H_w^{\beta,q}(\Omega)$, *where* $\theta = \frac{1+\beta}{2}$.

2.

$$\left[H_{w,0}^{-1,q}(\Omega), H_w^{1,q}(\Omega)\right]_\theta = \begin{cases} H_{w,0}^{\beta,q}(\Omega), & \text{if } \beta < 0 \\ H_w^{\beta,q}(\Omega), & \text{if } \beta \geq 0, \end{cases}$$

where $\theta = \frac{1+\beta}{2}$.

Proof. 1. $\{H_w^{-1,q}(\Omega), H_w^{1,q}(\Omega)\}$ is a retract of $\{H_w^{-1,q}(\mathbb{R}^n), H_w^{1,q}(\mathbb{R}^n)\}$ where the retraction is the restriction operator

$$R_\Omega : H_w^{\pm 1,q}(\mathbb{R}^n) \to H_w^{\pm 1,q}(\Omega), \quad u \mapsto u|_{C_0^\infty(\Omega)},$$

and the coretraction is the extension operator E constructed in Lemma 7.3.3. Thus the assertion in 1. follows from Theorem 2.3.1 and the corresponding interpolation property on $\mathbb{R}^n$ shown in Theorem 7.1.1
2. An application of the Duality Theorem 2.3.1 to 1. yields together with Lemma 7.3.2

$$\left[H_{w,0}^{-1,q}(\Omega), H_{w,0}^{1,q}(\Omega)\right]_\theta = H_{w,0}^{\beta,q}(\Omega). \tag{7.3.2}$$

Since, using the definition of the function class F as in (2.3) one obtains

$$F(H_{w,0}^{-1,q}(\Omega), H_{w,0}^{1,q}(\Omega)) \subset F(H_{w,0}^{-1,q}(\Omega), H_w^{1,q}(\Omega))$$

and the same is true when replacing q by q' and w by w', we have by (7.3.2)

$$L_w^q(\Omega) = \left[H_{w,0}^{-1,q}(\Omega), H_{w,0}^{1,q}(\Omega)\right]_{\frac{1}{2}} \hookrightarrow \left[H_{w,0}^{-1,q}(\Omega), H_w^{1,q}(\Omega)\right]_{\frac{1}{2}} \tag{7.3.3}$$

and

$$L_{w'}^{q'}(\Omega) \hookrightarrow \left[H_{w',0}^{-1,q'}(\Omega), H_{w'}^{1,q'}(\Omega)\right]_{\frac{1}{2}} = \left[H_{w,0}^{-1,q}(\Omega), H_w^{1,q}(\Omega)\right]_{\frac{1}{2}}' . \tag{7.3.4}$$

By the density of the embedding $H_{w'}^{1,q'}(\Omega) \hookrightarrow \left[H_{w',0}^{-1,q'}(\Omega), H_{w'}^{1,q'}(\Omega)\right]_{\frac{1}{2}}$ we obtain that the embedding (7.3.4) is dense. Thus we dualize (7.3.4) and combine it with (7.3.3) to obtain

$$\left[H_{w,0}^{-1,q}(\Omega), H_w^{1,q}(\Omega)\right]_{\frac{1}{2}} = L_w^q(\Omega).$$

Now the assertion follows by the reiteration property in Theorem 2.3.1 as follows. For $\beta < 0$ one uses (7.3.2) to obtain

$$\begin{aligned}
\left[H_{w,0}^{-1,q}(\Omega), H_w^{1,q}(\Omega)\right]_{\frac{1+\beta}{2}} &= \left[H_{w,0}^{-1,q}(\Omega), \left[H_{w,0}^{-1,q}(\Omega), H_w^{1,q}(\Omega)\right]_{\frac{1}{2}}\right]_{1+\beta} \\
&= \left[H_{w,0}^{-1,q}(\Omega), L_w^q(\Omega)\right]_{1+\beta} \\
&= \left[H_{w,0}^{-1,q}(\Omega), \left[H_{w,0}^{-1,q}(\Omega), H_{w,0}^{1,q}(\Omega)\right]_{\frac{1}{2}}\right]_{1+\beta} \\
&= \left[H_{w,0}^{-1,q}(\Omega), H_{w,0}^{1,q}(\Omega)\right]_{\frac{1+\beta}{2}} \\
&= H_{w,0}^{\beta,q}(\Omega).
\end{aligned}$$

Analogously, one shows the assertion for $\beta \geq 0$ replacing (7.3.2) by the assertion of 1. □

7.4 Interpolation of Bessel Potential Spaces with Zero Boundary Values

For an extension domain $\Omega \subset \mathbb{R}^n$, $1 < q < \infty$, $w \in A_q$ and $0 \le \beta \le 2$ we define the space

$$Y_w^{\beta,q}(\Omega) := \begin{cases} \overline{Y_w^{2,q}(\Omega)}^{H_w^{\beta,q}(\mathbb{R}^n)}, & \text{if } \ 0 \le \beta \le 1 \ \text{ equipped with } \|\cdot\|_{H_w^{\beta,q}(\mathbb{R}^n)}, \\ \overline{Y_w^{2,q}(\Omega)}^{H_w^{\beta,q}(\Omega)}, & \text{if } \ 1 < \beta \le 2 \ \text{ equipped with } \|\cdot\|_{H_w^{\beta,q}(\Omega)}, \end{cases}$$

where in the case $0 \le \beta \le 1$ the functions of $Y_w^{2,q}(\Omega)$ are assumed to be extended by 0 to functions defined on the whole space $\mathbb{R}^n$. This is possible, since $C_0^\infty(\Omega)$ is dense in $W_{w,0}^{1,q}(\Omega) \supset Y_w^{2,q}(\Omega)$ and $W_{w,0}^{1,q}(\Omega) \hookrightarrow W_w^{1,q}(\mathbb{R}^n) \hookrightarrow H_w^{\beta,q}(\mathbb{R}^n)$.

In particular, this implies that in the case $0 \le \beta \le 1$ one has

$$Y_w^{\beta,q}(\Omega) = \overline{C_0^\infty(\Omega)}^{H_w^{\beta,q}(\mathbb{R}^n)} = H_{w,0}^{\beta,q}(\Omega). \tag{7.4.1}$$

Moreover, for such β it follows immediately from the definition of $Y_w^{\beta,q}(\Omega)$ that the extension $E_0 u$ of functions $u \in Y_w^{\beta,q}(\Omega)$ by 0 to functions on $\mathbb{R}^n$ is a continuous linear map to $H_w^{\beta,q}(\mathbb{R}^n)$.

Finally, for $\beta = 1$ the two definitions are equivalent, i.e.,

$$Y_w^{1,q}(\Omega) = W_{w,0}^{1,q}(\Omega) = \overline{Y_w^{2,q}(\Omega)}^{H_w^{1,q}(\Omega)},$$

where the latter space is equipped with $\|\cdot\|_{H_w^{1,q}(\Omega)}$. The reason is that for $u \in Y_w^{2,q}(\Omega)$ one has by Theorem 7.1.2

$$\begin{aligned} \|u\|_{H_w^{1,q}(\Omega)} &\le c_1\|u\|_{W_w^{1,q}(\Omega)} = c_1\|E_0u\|_{W_w^{1,q}(\mathbb{R}^n)} \le c_2\|E_0u\|_{H_w^{1,q}(\mathbb{R}^n)} \\ &= c_2\|u\|_{Y_w^{1,q}(\Omega)} \le c_3\|u\|_{H_w^{1,q}(\Omega)}, \end{aligned}$$

where E_0 stands for the extension by 0. For symmetry reasons the question arises whether $Y_w^{\beta,q}(\Omega) = \overline{Y_w^{2,q}(\Omega)}^{H_w^{\beta,q}(\Omega)}$ for all $0 \le \beta \le 2$. However this is not the case, not even in the unweighted case. Indeed, by Triebel [53, I.5.23] one has

$$\begin{aligned} \overline{Y_1^{2,q}(\Omega)}^{H^{\frac{1}{q},q}(\Omega)} &= \overline{C_0^\infty(\Omega)}^{H^{\frac{1}{q},q}(\Omega)} \\ &\ne \{u \in H^{\frac{1}{q},q}(\mathbb{R}^n) \mid \operatorname{supp} u \subset \overline{\Omega}\} = Y_1^{\frac{1}{q},q}(\Omega). \end{aligned} \tag{7.4.2}$$

We choose the spaces $Y_w^{\beta,q}(\Omega)$ because of their good properties with respect to interpolation.

Theorem 7.4.1. *Let $1 < q < \infty$, $w \in A_q$ and $0 \le \beta \le 2$. Then*

$$\left[L_w^q(\mathbb{R}_+^n), Y_w^{2,q}(\mathbb{R}_+^n)\right]_\theta = Y_w^{\beta,q}(\mathbb{R}_+^n), \quad \theta = \frac{\beta}{2}$$

with equivalent norms.

Proof. We may assume that $w = w^*$ (given by (3.1.3)), i.e. w is even in x_n. This can be done because the norm in $Y_w^{\beta,q}(\mathbb{R}_+^n)$ is equivalent to the one in $Y_{\tilde{w}}^{\beta,q}(\mathbb{R}_+^n)$ if $\tilde{w} \in A_q$ with $\tilde{w}|_{\mathbb{R}_+^n} = w|_{\mathbb{R}_+^n}$. In the case $\beta \geq 1$ this is true by Theorem 7.1.2.

If $\beta < 1$ one has by Theorem 7.3.4 and (7.4.1)

$$Y_w^{\beta,q}(\mathbb{R}_+^n) = H_{w,0}^{\beta,q}(\mathbb{R}_+^n) = \left(H_{w'}^{-\beta,q'}(\mathbb{R}_+^n)\right)' = [H_{w'}^{1,q'}(\mathbb{R}_+^n), H_{w'}^{-1,q'}(\mathbb{R}_+^n)]'_{\frac{\beta+1}{2}}.$$

The latter interpolation space is independent of the weight function outside $\mathbb{R}_+^n$, because $H_{w'}^{1,q'}(\mathbb{R}_+^n)$ and $H_{w'}^{-1,q'}(\mathbb{R}_+^n)$ are.

Step 1: We show that

$$\left[L_w^q(\mathbb{R}_+^n), Y_w^{2,q}(\mathbb{R}_+^n)\right]_\theta \hookrightarrow Y_w^{\beta,q}(\mathbb{R}_+^n).$$

To see this let $u \in \left[L_w^q(\mathbb{R}_+^n), Y_w^{2,q}(\mathbb{R}_+^n)\right]_\theta$.

We begin with the case $1 \leq \beta \leq 2$. Then there is a function $U \in F(L_w^q(\mathbb{R}_+^n), Y_w^{2,q}(\mathbb{R}_+^n))$ such that $U(\theta) = u$ and $\|U\|_{F(L_w^q(\mathbb{R}_+^n), Y_w^{2,q}(\mathbb{R}_+^n))} \leq 2\|u\|_{[L_w^q(\mathbb{R}_+^n), Y_w^{2,q}(\mathbb{R}_+^n)]_\theta}$.

Since $F(L_w^q(\mathbb{R}_+^n), Y_w^{2,q}(\mathbb{R}_+^n)) \subset F(L_w^q(\mathbb{R}_+^n), H_w^{2,q}(\mathbb{R}_+^n))$, we obtain

$$u = U(\theta) \in [L_w^q(\mathbb{R}_+^n), H_w^{2,q}(\mathbb{R}_+^n)]_\theta = H_w^{\beta,q}(\mathbb{R}_+^n)$$

and

$$\begin{aligned}
\|u\|_{H_w^{\beta,q}(\mathbb{R}_+^n)} &\leq c \inf\left\{\|V\|_{F(L_w^q(\mathbb{R}_+^n), H_w^{2,q}(\mathbb{R}_+^n))} \;\middle|\; V \in F(L_w^q(\mathbb{R}_+^n), H_w^{2,q}(\mathbb{R}_+^n)),\ V(\theta) = u\right\} \\
&\leq c\|U\|_{F(L_w^q(\mathbb{R}_+^n), Y_w^{2,q}(\mathbb{R}_+^n))} \leq 2\|u\|_{[L_w^q(\mathbb{R}_+^n), Y_w^{2,q}(\mathbb{R}_+^n)]_\theta}.
\end{aligned}$$

Moreover, by Theorem 2.3.1 we know that $Y_w^{2,q}(\mathbb{R}_+^n)$ is dense in $[L_w^q(\mathbb{R}_+^n), Y_w^{2,q}(\mathbb{R}_+^n)]_\theta$ which yields the assertion of Step 1 in the case $\beta \geq 1$.

In the case $0 \leq \beta \leq 1$ we assume that we already know $\left[L_w^q(\mathbb{R}_+^n), Y_w^{2,q}(\mathbb{R}_+^n)\right]_{\frac{1}{2}} = Y_w^{1,q}(\mathbb{R}_+^n)$. This follows from the case $1 \leq \beta \leq 2$ which will be shown independently. Then, since

$$Y_w^{1,q}(\mathbb{R}_+^n) = \overline{C_0^\infty(\mathbb{R}_+^n)}^{W_w^{1,q}(\mathbb{R}^n)} = W_{w,0}^{1,q}(\mathbb{R}_+^n),$$

the reiteration property implies for $0 \leq \theta \leq \frac{1}{2}$

$$\left[L_w^q(\mathbb{R}_+^n), Y_w^{2,q}(\mathbb{R}_+^n)\right]_\theta = \left[L_w^q(\mathbb{R}_+^n), W_{w,0}^{1,q}(\mathbb{R}_+^n)\right]_{2\theta}.$$

Since the extension

$$E_0 u(x) = \begin{cases} u(x) & \text{for } x \in \mathbb{R}_+^n \\ 0 & \text{for } x \in \mathbb{R}_-^n \end{cases}$$

of functions defined on the half space is continuous from $W_{w,0}^{1,q}(\mathbb{R}_+^n)$ to $W_w^{1,q}(\mathbb{R}^n)$ and from $L_w^q(\mathbb{R}_+^n)$ to $L_w^q(\mathbb{R}^n)$, we find by interpolation that

$$E_0 : \left[L_w^q(\mathbb{R}_+^n), W_{w,0}^{1,q}(\mathbb{R}_+^n)\right]_{2\theta} \to H_w^{\beta,q}(\mathbb{R}^n)$$

is continuous. Thus for every $u \in C_0^\infty(\mathbb{R}_+^n)$ we obtain

$$\|u\|_{Y_w^{\beta,q}(\mathbb{R}_+^n)} = \|E_0 u\|_{\beta,q,w,\mathbb{R}^n} \leq c\|u\|_{[L_w^q(\mathbb{R}_+^n), W_{w,0}^{1,q}(\mathbb{R}_+^n)]_{2\theta}}.$$

Then the density of the embedding $C_0^\infty(\mathbb{R}^n_+) \hookrightarrow \left[L^q_w(\mathbb{R}^n_+), W^{1,q}_{w,0}(\mathbb{R}^n_+)\right]_{2\theta}$ finishes the proof of Step 1.
Step 2: Claim: If the odd extension,

$$E_{odd} : Y^{\beta,q}_w(\mathbb{R}^n_+) \to H^{\beta,q}_w(\mathbb{R}^n),$$

is continuous, where

$$E_{odd}u(x) = \begin{cases} u(x) & \text{if } x \in \mathbb{R}^n_+ \\ -u(x', -x_n) & \text{if } x \in \mathbb{R}^n_- \end{cases}$$

for $x = (x', x_n)$, then the assertion of $Y^{\beta,q}_w(\mathbb{R}^n_+) \hookrightarrow \left[L^q_w(\mathbb{R}^n_+), Y^{2,q}_w(\mathbb{R}^n_+)\right]_\theta$ is true for β.
Proof of the Claim. Let $u \in Y^{\beta,q}_w(\mathbb{R}^n_+)$ and set

$$U(z) = e^{z^2}\Lambda^{(\theta-z)2}E_{odd}u.$$

Then one has $U \in F(L^q_w(\mathbb{R}^n), W^{2,q}_w(\mathbb{R}^n))$ with $U(\theta) = e^{\theta^2}E_{odd}u$. Moreover, since for every $\mu \in \mathbb{C}$ the operator Λ^μ maps odd functions to odd functions, one has $U(iy+1)|_{\mathbb{R}^{n-1}} = 0$ which implies $U(iy+1)|_{\mathbb{R}^n_+} \in Y^{2,q}_w(\mathbb{R}^n_+)$ for every y. Thus $U|_{\mathbb{R}^n_+} \in F(L^q_w(\mathbb{R}^n_+), Y^{2,q}_w(\mathbb{R}^n_+))$ and we obtain $u \in \left[L^q_w(\mathbb{R}^n_+), Y^{2,q}_w(\mathbb{R}^n_+)\right]_\theta$ with

$$\begin{aligned}
\|u\|_{[L^q_w(\mathbb{R}^n_+), Y^{2,q}_w(\mathbb{R}^n_+)]_\theta} &\leq \sup_y \|U(iy+1)\|_{Y^{2,q}_w(\mathbb{R}^n_+)} + \sup_y \|U(iy)\|_{L^q_w(\mathbb{R}^n_+)} \\
&\leq 2\left(\sup_y \|U(iy+1)\|_{Y^{2,q}_w(\mathbb{R}^n)} + \sup_y \|U(iy)\|_{L^q_w(\mathbb{R}^n)}\right) \\
&\leq c\|E_{odd}u\|_{H^{\beta,q}_w(\mathbb{R}^n)} \\
&\leq c\|u\|_{Y^{\beta,q}_w(\mathbb{R}^n_+)}.
\end{aligned}$$

Step 3: The embedding $Y^{\beta,q}_w(\mathbb{R}^n_+) \hookrightarrow \left[L^q_w(\mathbb{R}^n_+), Y^{2,q}_w(\mathbb{R}^n_+)\right]_\theta$ is true for $\beta < 1$.
By the definition of $Y^{\beta,q}_w(\mathbb{R}^n_+)$ for $\beta < 1$ we know that the extension E_0u of u by 0 on $\mathbb{R}^n$ is continuous from $Y^{\beta,q}_w(\mathbb{R}^n_+)$ to $H^{\beta,q}_w(\mathbb{R}^n)$ with norm 1. Thus the odd extension of u, which is equal to

$$E_{odd}u(x) = E_0u(x) - E_0u(x', -x_n),$$

is also continuous. Step 2 completes the argument.
Step 4: The embedding $Y^{\beta,q}_w(\mathbb{R}^n_+) \hookrightarrow \left[L^q_w(\mathbb{R}^n_+), Y^{2,q}_w(\mathbb{R}^n_+)\right]_\theta$ is true for $1 \leq \beta \leq 2$.
For $g \in T^{2,q}_w(\mathbb{R}^{n-1})$ there exists an extension $S(g)$ with the following properties:

- $S(g)|_{\mathbb{R}^{n-1}} = g$.
- S is a continuous linear mapping

$$S : T^{2,q}_w(\mathbb{R}^{n-1}) \to W^{2,q}_w(\mathbb{R}^n) \quad \text{and} \quad S : T^{1,q}_w(\mathbb{R}^{n-1}) \to W^{1,q}_w(\mathbb{R}^n).$$

To see this we define $S(g)|_{\mathbb{R}^n_+}$ to be the solution of

$$(1-\Delta)S(g) = 0 \quad \text{on } \mathbb{R}^n_+ \quad \text{and} \quad S(g) = g \quad \text{on } \mathbb{R}^{n-1}.$$

Then by [25, Lemma 3.14, Satz 3.7] we know that $S(g)|_{\mathbb{R}^n_+}$ is well-defined and has the two properties on $\mathbb{R}^n_+$. By Theorem 3.3.2 there exists an extension operator, continuous from $W^{2,q}_w(\mathbb{R}^n_+)$ to $W^{2,q}_w(\mathbb{R}^n)$ as well as from $W^{1,q}_w(\mathbb{R}^n_+)$ to $W^{1,q}_w(\mathbb{R}^n)$. Thus the existence of such an S is proved.

Now we consider the operator

$$B : H^{2,q}_w(\mathbb{R}^n_+) \to H^{2,q}_w(\mathbb{R}^n), \quad u \mapsto S(u|_{\mathbb{R}^{n-1}}) + E_{odd}(u - S(u|_{\mathbb{R}^{n-1}})).$$

Since $w = \tilde{w}$ and $Y^{2,q}_w(\mathbb{R}^n_+)|_{\mathbb{R}^{n-1}} = \{0\}$, it is easy to check that the operator E_{odd} is continuous from $Y^{2,q}_w(\mathbb{R}^n_+)$ to $W^{2,q}_w(\mathbb{R}^n)$ and from $W^{1,q}_{w,0}(\mathbb{R}^n_+)$ to $W^{1,q}_w(\mathbb{R}^n)$. Thus, we have constructed an operator B which is continuous from $W^{2,q}_w(\mathbb{R}^n_+)$ to $W^{2,q}_w(\mathbb{R}^n)$ as well as from $W^{1,q}_w(\mathbb{R}^n_+)$ to $W^{1,q}_w(\mathbb{R}^n)$ and which coincides with E_{odd} on $Y^{\beta,q}_w(\mathbb{R}^n_+)$, $\beta = 1, 2$. By interpolation we find that

$$B : H^{\beta,q}_w(\mathbb{R}^n_+) \to H^{\beta,q}_w(\mathbb{R}^n)$$

is continuous for every $1 \le \beta \le 2$. Thus for every $u \in Y^{\beta,q}_w(\mathbb{R}^n_+) \subset Y^{1,q}_w(\mathbb{R}^n_+)$ one has

$$\|E_{odd}u\|_{H^{\beta,q}_w(\mathbb{R}^n)} = \|Bu\|_{H^{\beta,q}_w(\mathbb{R}^n)} \le c\|u\|_{H^{\beta,q}_w(\mathbb{R}^n_+)} = c\|u\|_{Y^{\beta,q}_w(\mathbb{R}^n_+)}.$$

Thus Step 2 finishes the proof. □

Theorem 7.4.2. *The assertion of Theorem 7.4.1 holds true, when replacing* $\mathbb{R}^n_+$ *by a bounded* $C^{1,1}$*-domain* Ω*, i.e.,*

$$\left[L^q_w(\Omega), Y^{2,q}_w(\Omega)\right]_\theta = Y^{\beta,q}_w(\Omega), \quad \theta = \frac{\beta}{2}, \quad 0 \le \beta \le 2$$

with equivalent norms.

Proof. Let α_j, $j = 1, ..., m$, be a collection of $C^{1,1}$-charts and ψ_j a decomposition of unity subordinate to the corresponding covering of $\overline{\Omega}$. We assume that every ψ_j is extended to an element of $C^\infty_0(\mathbb{R}^n)$ and that every α_i is extended to an element of $C^{1,1}(\mathbb{R}^n)$ such that it has an inverse $\alpha_j^{-1} \in C^{1,1}(\mathbb{R}^n)$.

Then we fix j, write $\psi = \psi_j$ and $\alpha = \alpha_j$ and define the mapping

$$B : Y^{\beta,q}_{w\circ\alpha}(\mathbb{R}^n_+) \to Y^{\beta,q}_w(\Omega), \quad u \mapsto (u \cdot (\psi \circ \alpha)) \circ \alpha^{-1}. \tag{7.4.3}$$

We have to show, that B is a continuous mapping into the asserted image space.
Case 1 ($0 \le \beta \le 1$): In this case the extension $E_0 : Y^{\beta,q}_{w\circ\alpha}(\mathbb{R}^n_+) \to H^{\beta,q}_{w\circ\alpha}(\mathbb{R}^n)$ by zero is continuous. The operator

$$\overline{B} : H^{\beta,q}_{w\circ\alpha}(\mathbb{R}^n) \to H^{\beta,q}_w(\mathbb{R}^n), \quad u \mapsto (u(\psi \circ \alpha)) \circ \alpha^{-1}$$

is continuous by Lemma 7.2.1. Thus we obtain for $u \in Y^{\beta,q}_{w\circ\alpha}(\mathbb{R}^n_+)$

$$\|Bu\|_{Y^{\beta,q}_w(\Omega)} = \|E_0 Bu\|_{\beta,q,w,\mathbb{R}^n} = \|\overline{B}E_0 u\|_{\beta,q,w,\mathbb{R}^n} \le c\|E_0 u\|_{\beta,q,w\circ\alpha,\mathbb{R}^n} = c\|u\|_{Y^{\beta,q}_{w\circ\alpha}(\mathbb{R}^n_+)}.$$

Thus we obtain the continuity of B in (7.4.3) for $0 \le \beta \le 1$.
Case 2 ($1 \le \beta \le 2$): Let R_Ω denote the restriction of functions defined on $\mathbb{R}^n$ to Ω.

Interpolation shows that $R_\Omega \circ B$, defined in the same way for functions on $\mathbb{R}^n$, maps $H^{\beta,q}_{w\circ\alpha}(\mathbb{R}^n)$ continuously to $H^{\beta,q}_w(\Omega)$. Since

$$(R_\Omega \circ B)(\{u \in W^{2,q}_{w\circ\alpha}(\mathbb{R}^n) \mid u|_{\mathbb{R}^{n-1}} = 0\}) \subset \{u \in W^{2,q}_w(\Omega) \mid u|_{\partial\Omega} = 0\},$$

the operator $B : Y^{\beta,q}_{w\circ\alpha}(\mathbb{R}^n_+) \to Y^{\beta,q}_w(\Omega)$ is continuous by the density of $Y^{2,q}_w(\Omega)$ in $Y^{\beta,q}_w(\Omega)$. This proves (7.4.3) for $\beta \in (1,2]$.

Now setting $B_j u = (u(\psi_j \circ \alpha_j)) \circ \alpha_j^{-1}$ we define the operator

$$B_\Omega : \prod_{i=1}^m Y^{\beta,q}_{w\circ\alpha_i}(\mathbb{R}^n_+) \to Y^{\beta,q}_w(\Omega), \quad (u_1, ..., u_m) \mapsto \sum_{i=1}^m B_i u_i,$$

which is continuous and surjective for every $\beta \in [0,2]$. (Surjectivity follows if one considers the operator

$$A_j : H^{\beta,q}_w(\Omega) \ni u \mapsto (u\phi_j) \circ \alpha_j \in H^{\beta,q}_{w\circ\alpha_j}(\mathbb{R}^n_+), \quad j = 1, ..., m,$$

where ϕ_j is an appropriate cut-off function, with $\phi_j \equiv 1$ on $\operatorname{supp} \psi_j$.)

Moreover, by interpolation and Theorem 7.4.1 it follows that

$$B_\Omega : \prod_{i=1}^m Y^{\beta,q}_{w_i}(\mathbb{R}^n_+) \to \left[L^q_w(\Omega), Y^{2,q}_w(\Omega)\right]_{\frac{\beta}{2}}$$

is continuous, where $w_i := w \circ \alpha_i$.

For every $u \in Y^{\beta,q}_w(\Omega)$ there exists $(u_1, ..., u_m) \in \prod_{i=1}^m Y^{\beta,q}_{w_i}(\mathbb{R}^n_+)$ with $B_\Omega(u_1, ..., u_m) = u$ and $\|u_i\|_{Y^{\beta,q}_{w_i}(\mathbb{R}^n_+)} \leq c\|u\|_{Y^{\beta,q}_w(\Omega)}$ for every $i = 1, ..., m$. Then one can estimate

$$\begin{aligned}\|u\|_{[L^q_w(\Omega), Y^{2,q}_w(\Omega)]_\theta} =& \|B_\Omega(u_1, ..., u_n)\|_{[L^q_w(\Omega), Y^{2,q}_w(\Omega)]_\theta} \\ \leq& c\sum_{i=1}^m \|u_i\|_{[L^q_{w_i}(\mathbb{R}^n_+), Y^{2,q}_{w_i}(\mathbb{R}^n_+)]_\theta} \leq c\sum_{i=1}^m \|u_i\|_{Y^{\beta,q}_{w_i}(\mathbb{R}^n_+)} \\ \leq& c\|u\|_{Y^{\beta,q}_w(\Omega)}.\end{aligned}$$

Thus we obtain $[L^q_w(\Omega), Y^{2,q}_w(\Omega)]_{\frac{\beta}{2}} \supset Y^{\beta,q}_w(\Omega)$.

The inclusion "$\subset$" is proved in the same way as in the proof of Theorem 7.4.1, Step 1. $\square$

8 Stokes Equations in Bessel Potential Spaces

The reason why we consider solutions to the Stokes equations in weighted Bessel potential spaces is pointed out in Chapter 10 below. There we are dealing with the stationary Navier-Stokes equations. The estimates of the nonlinear term require strong assumptions on the weight function. One possibility to avoid this is to work in spaces of higher regularity.

In Section 8.1 existence and uniqueness of solutions to the stationary Stokes equations are proved. The basic tool is interpolation between the strong and the very weak solutions. Therefore, the results from Chapter 7 are frequently used.

The results from Section 8.1 lead to a a generalization of the Stokes operator in Section 8.2. This operator is appropriate in the context of very weak solutions in weighted Bessel potential spaces. Moreover it possesses many properties as the classical Stokes operator. In particular, it generates an analytic semigroup and has maximal regularity. These results are of great importance when dealing with the instationary case in Chapter 9.

8.1 Stokes Equations in Weighted Bessel Potential Spaces

Let $\beta \in [0,2]$, $q \in (1,\infty)$ and $w \in A_q$. Our aim is to obtain solutions to the Stokes equations in $H^{\beta,q}_w(\Omega)$ presumed the data is sufficiently regular.

As a space for exterior forces we define

$$f \in Y^{-\beta,q}_w(\Omega) := \left(Y^{\beta,q'}_{w'}(\Omega)\right)'.$$

Note that if $0 \le \beta \le 1$ then by (7.4.1) one has the embedding

$$Y^{-\beta,q}_w(\Omega) = H^{-\beta,q}_w(\Omega) \hookrightarrow W^{-1,q}_w(\Omega)$$

and thus $Y^{-\beta,q}_w(\Omega)$ consists of distributions on Ω.

If $\beta > 1$ then this is in general not the case. In particular, if β is big enough, then a functional $f \in Y^{-\beta,q}_w(\Omega)$ might include a part supported on the boundary which can be considered as a boundary condition.

As a space for divergences we choose

$$H^{\gamma,q}_{w,*}(\Omega) := \begin{cases} H^{\gamma,q}_w(\Omega), & \text{if } \gamma \ge 0 \\ H^{\gamma,q}_{w,0}(\Omega), & \text{if } \gamma < 0, \end{cases}$$

for every $\gamma \in [-1,1]$. This space is equipped with the norm $\|\cdot\|_{\gamma,q,w,*,\Omega} := \|\cdot\|_{H^{\gamma,q}_{w,*}(\Omega)}$.

As in Chapter 6 we call $u \in H^{\beta,q}_w(\Omega)$ a very weak solution to the Stokes equation, if

$$\begin{aligned} \langle f,\varphi\rangle &= -\langle u,\Delta\varphi\rangle, &&\text{for all } \varphi \in Y^{2,q'}_{w',\sigma}(\Omega) \text{ and} \\ \langle k,\psi\rangle &= -\langle u,\nabla\psi\rangle, &&\text{for all } \psi \in W^{1,q'}_{w'}(\Omega). \end{aligned} \tag{8.1.1}$$

If $\beta \geq 1$, i.e., the solution is contained in $W^{1,q}_w(\Omega)$, then we also refer to the very weak solutions as weak solutions.

Theorem 8.1.1. *Let $1 < q < \infty$, $w \in A_q$, $0 \leq \beta \leq 2$ and let Ω be a bounded $C^{1,1}$-domain. Moreover, let*

$$f \in Y^{\beta-2,q}_w(\Omega) \quad \textit{and} \quad k \in H^{\beta-1,q}_{w,*}(\Omega)$$

with $\langle k,1\rangle = 0$. Then there exists a unique very weak solution $u \in Y^{\beta,q}_w(\Omega)$ to the Stokes problem with respect to the data f, k in the sense of (8.1.1). This function u fulfills the estimate

$$\|u\|_{Y^{\beta,q}_w(\Omega)} \leq c\left(\|f\|_{Y^{\beta-2,q}_w(\Omega)} + \|k\|_{\beta-1,q,w,*,\Omega}\right).$$

Moreover, there exists a pressure functional $p \in H^{\beta-1,q}_w(\Omega)$, unique modulo constants, such that

$$-\Delta u + \nabla p = f|_{C_0^\infty(\Omega)} \quad \textit{in} \quad C_0^\infty(\Omega)'.$$

Proof. From the results in Sections 7.3 and 7.4 it follows that

$$\left[Y^{-2,q}_w(\Omega)\times H^{-1,q}_{w,0}(\Omega), L^q_w(\Omega)\times H^{1,q}_w(\Omega)\right]_\theta = Y^{\beta-2,q}_w(\Omega)\times H^{\beta-1,q}_{w,*}(\Omega),$$

where $\theta = \frac{\beta}{2}$. It is immediate that

$$k \mapsto K := k - \langle k,1\rangle \in \mathcal{L}(H^{-1,q}_{w,0}(\Omega)) \cap \mathcal{L}(H^{1,q}_w(\Omega)).$$

By Theorem 6.1.4 the mapping

$$\mathcal{S} : Y^{-2,q}_w(\Omega)\times H^{-1,q}_{w,0}(\Omega) \ni (f,k) \mapsto u \in L^q_w(\Omega),$$

is continuous, where $u \in L^q_w(\Omega)$ is the very weak solution to the Stokes problem with respect to the data f and $K = k - \langle k,1\rangle$.

If u is a solution to (8.1.1) with sufficiently regular data f and k, then by Theorem 6.2.2 we find that u is a strong solution with zero boundary values. In particular, $\mathcal{S}$ is also continuous from $L^q_w(\Omega)\times H^{1,q}_w(\Omega)$ to $Y^{2,q}_w(\Omega)$. Now we obtain from interpolation that

$$\mathcal{S} : Y^{\beta-2,q}_w(\Omega)\times H^{\beta-1,q}_{w,*}(\Omega) \to Y^{\beta,q}_w(\Omega)$$

is continuous, which finishes the proof of existence and estimates of u. Uniqueness follows from the uniqueness of very weak solutions in $L^q_w(\Omega)$ (Theorem 6.1.4).

It remains to show the existence of p. By the theory of strong solutions in [27] there exists a unique (modulo constants) pressure function $p \in H^{1,q}_w(\Omega)$. Moreover, by Theorem 6.1.4 there exists a pressure functional $p \in H^{-1,q}_{w,0}(\Omega)$ that belongs to a very weak solution. Thus by the interpolation Theorem 7.3.4.2 we obtain a functional $\tilde{p} \in H^{\beta-1,q}_{w,*}(\Omega)$ such that

$$-\langle u,\Delta\phi\rangle - \langle\tilde{p},\operatorname{div}\phi\rangle = \langle F,\phi\rangle \quad \text{for all} \quad \phi \in Y^{2,q'}_{w'}(\Omega).$$

The restriction $p := \tilde{p}|_{C_0^\infty(\Omega)}$ solves the problem. $\square$

By the definition of $Y_w^{\beta,q}(\Omega)$ it follows, that whenever a trace operator

$$\mathrm{tr}\, : H_w^{\beta,q}(\Omega) \to T(D)$$

for a boundary portion $D \subset \partial\Omega$ is well-defined (as a continuous linear operator into some boundary space $T(D)$, which coincides with the usual trace $u|_D$ on $W_w^{1,q}(\Omega)$), then for the solution $u \in Y_w^{\beta,q}(\Omega)$ one has $\mathrm{tr}\, u = 0$.

In the case, where data and solutions are regular enough (including the case $\beta = 1$ of weak solutions), we want to deal with inhomogeneous boundary values.

If $\beta \geq 1$, then $H_w^{\beta,q}(\Omega) \hookrightarrow W_w^{1,q}(\Omega)$ which implies the existence of a continuous trace operator

$$\mathrm{tr}\, : H_w^{\beta,q}(\Omega) \to T_w^{1,q}(\partial\Omega), \quad \mathrm{tr}\, u = u|_{\partial\Omega} \text{ if } \quad u \in C^\infty(\overline{\Omega}).$$

As in the case of weighted Sobolev spaces we define the associated boundary space by

$$T_w^{\beta,q}(\partial\Omega) = \mathrm{tr}\, \left(H_w^{\beta,q}(\Omega)\right)$$

equipped with the norm of the factor space

$$\|g\|_{T_w^{\beta,q}(\partial\Omega)} = \inf\left\{\|u\|_{\beta,q,w,\Omega} \mid u \in H_w^{\beta,q}(\Omega),\ \mathrm{tr}\, u = g\right\}.$$

Lemma 8.1.2. *For every $\beta \in [1,2]$ one has*

$$[T_w^{1,q}(\partial\Omega), T_w^{2,q}(\partial\Omega)]_{\beta-1} = T_w^{\beta,q}(\partial\Omega)$$

and there exists a continuous linear extension operator

$$\mathrm{ext}\, : T_w^{\beta,q}(\partial\Omega) \to H_w^{\beta,q}(\Omega),$$

independent of β.

Proof. By Theorem 4.1.2 there exists a continuous linear extension operator

$$\mathrm{ext}\, : T_w^{1,q}(\partial\Omega) \to W_w^{1,q}(\Omega) \text{ and } \mathrm{ext}\, : T_w^{2,q}(\partial\Omega) \to W_w^{2,q}(\Omega), \tag{8.1.2}$$

where $\mathrm{ext}\, g$ is given by the solution of $(1-\Delta)(\mathrm{ext}\, g) = 0$ and $(\mathrm{ext}\, g)|_{\partial\Omega} = g$ for every $g \in T_w^{k,q}(\partial\Omega)$, $k = 1, 2$.

Moreover, by definition the trace operator tr is continuous

$$\mathrm{tr}\, : W_w^{1,q}(\Omega) \to T_w^{1,q}(\partial\Omega) \text{ and } \mathrm{tr}\, : W_w^{2,q}(\Omega) \to T_w^{2,q}(\partial\Omega).$$

Obviously one has $\mathrm{tr} \circ \mathrm{ext} = \mathrm{id}_{T_w^{1,q}(\partial\Omega)}$ and thus Theorem 2.3.1.5 shows

$$[T_w^{1,q}(\partial\Omega), T_w^{2,q}(\partial\Omega)]_{\beta-1} = \mathrm{tr}\,[W_w^{1,q}(\Omega), W_w^{2,q}(\Omega)]_{\beta-1} = \mathrm{tr}\, H_w^{\beta,q}(\Omega) = T_w^{\beta,q}(\partial\Omega).$$

Thus the first assertion is proved. The second assertion follows from the first combined with (8.1.2). □

Theorem 8.1.3. *Let $1 < q < \infty$, $w \in A_q$ and $1 \le \beta \le 2$. Moreover, let $F \in H_w^{\beta-2,q}(\Omega)$, $K \in H_w^{\beta-1,q}(\Omega)$ and $g \in T_w^{\beta,q}(\partial\Omega)$ with $\int_\Omega K = \int_{\partial\Omega} g \cdot N$. Then there exists a unique weak solution $u \in H_w^{\beta,q}(\Omega)$, i.e.,*

$$(\nabla u, \nabla\phi) = \langle F, \phi\rangle, \quad \textit{for all} \ \ \phi \in W_{w,0,\sigma}^{1,q}(\Omega)$$

fulfilling $u|_{\partial\Omega} = g$ and $\operatorname{div} u = K$ in the sense of distributions. This solution fulfills the estimate

$$\|u\|_{\beta,q,w} \le c\left(\|F\|_{\beta-2,q,w} + \|K\|_{\beta-1,q,w} + \|g\|_{T_w^{\beta,q}(\partial\Omega)}\right).$$

Moreover, there exists a pressure function $p \in H_w^{\beta-1,q}(\Omega)$, unique modulo constants, such that the Stokes equations are fulfilled in the sense of distributions.

Proof. First of all recall that if $\beta \in [1,2]$, then $\beta - 2 \in [-1,0]$, which implies

$$F \in H_w^{\beta-2,q}(\Omega) = Y_w^{\beta-2,q}(\Omega).$$

Existence: For $g \in T_w^{\beta,q}(\partial\Omega)$ there exists $v \in H_w^{\beta,q}(\Omega)$ such that $\operatorname{tr} v = g$ and $\|v\|_{\beta,q,w,\Omega} \le 2\|g\|_{T_w^{\beta,q}(\partial\Omega)}$. Since there exists an extension V of v to the whole space $\mathbb{R}^n$ that fulfills the estimate $\|V\|_{\beta,q,w,\mathbb{R}^n} \le c\|v\|_{\beta,q,w,\Omega}$, one has

$$\Delta v = (\Delta V)|_{C_0^\infty(\Omega)} \in H_w^{\beta-2,q}(\Omega) = Y_w^{\beta-2,q}(\Omega).$$

Hence by Theorem 8.1.1 there exists $U \in H_w^{\beta,q}(\Omega)$ solving

$$\begin{aligned} \langle F + \Delta v, \phi\rangle &= -\langle U, \Delta\phi\rangle \qquad \text{for all } \phi \in Y_{w',\sigma}^{2,q'}(\Omega) \ \text{ and} \\ \langle K - \operatorname{div} v, \psi\rangle &= -\langle U, \nabla\psi\rangle \qquad \text{for all } \psi \in W_{w'}^{1,q'}(\Omega). \end{aligned}$$

Since $U \in Y_w^{\beta,q}(\Omega) \subset W_{w,0}^{1,q}(\Omega)$, we obtain by integration by parts for $\phi \in Y_{w',\sigma}^{2,q'}(\Omega)$, which is dense in $W_{w',0}^{1,q'}(\Omega)$, that

$$(\nabla(U+v), \nabla\phi) = -(U, \Delta\phi) - \langle \Delta v, \phi\rangle = \langle F, \phi\rangle,$$

where by the density of $C_0^\infty(\Omega)$ in $W_{w',0}^{1,q'}(\Omega)$ one can apply the definition of the derivatives div to ∇v in the sense of distributions. Setting $u := U + v$ we obtain $\operatorname{div} u = K$ in the sense of distributions and

$$\operatorname{tr} u = \operatorname{tr} v + \operatorname{tr} U = \operatorname{tr} v = g.$$

Moreover,

$$\begin{aligned} \|u\|_{\beta,q,w,\Omega} \le& \|v\|_{\beta,q,w,\Omega} + \|U\|_{\beta,q,w,\Omega} \\ \le& c\big(\|g\|_{T_w^{\beta,q}(\partial\Omega)} + \|F\|_{\beta-2,q,w,\Omega} + \|\Delta v\|_{\beta-2,q,w,\Omega} \\ &\quad + \|K\|_{\beta-1,q,w,\Omega} + \|\operatorname{div} v\|_{\beta-1,q,w,\Omega}\big) \\ \le& c\left(\|g\|_{T_w^{\beta,q}(\partial\Omega)} + \|F\|_{\beta-2,q,w,\Omega} + \|K\|_{\beta-1,q,w,\Omega}\right). \end{aligned}$$

Uniqueness: Let u be a weak solution to the Stokes problem with respect to the data F, K and g. Then integration by parts yields for every $\phi \in Y_{w',\sigma}^{2,q'}(\Omega)$

$$(u, \Delta\phi) = -(\nabla u, \nabla\phi) + \langle u, N \cdot \nabla\phi\rangle_{\partial\Omega} = -\langle F, \phi\rangle + \langle g, N \cdot \nabla\phi\rangle_{\partial\Omega}. \tag{8.1.3}$$

Since $|\langle g, N\cdot\nabla\phi\rangle_{\partial\Omega}| \le c\|g\|_{T_w^{0,q}(\partial\Omega)}\|\phi\|_{2,q',w',\Omega}$, we find that the right hand side of (8.1.3), considered as a map in ϕ, is contained in $Y_w^{-2,q}(\Omega)$. Thus u is a very weak solution. By the uniqueness of very weak solutions in Theorem 6.1.4, we obtain the uniqueness of u.
Pressure: To show the existence of p we use that by de Rham's Theorem [51, Ch.1 Proposition 1.1] there exists $p \in (C_0^\infty(\Omega))'$ such that the Stokes equations are fulfilled in the sense of distributions. From the equation we obtain $\nabla p \in H_w^{\beta-2,q}(\Omega)$. It remains to show $p \in H_w^{\beta-1,q}(\Omega)$. However, this follows by Lemma 8.1.7 below and the proof is complete □

Now we turn to the case $0 \le \beta \le 1$. In this case the functions in $H_w^{\beta,q}(\Omega)$ in general do not possess enough regularity to guarantee the well-definedness of a trace operator. Here we define boundary spaces by

$$T_w^{\beta,q}(\partial\Omega) = \left[T_w^{0,q}(\partial\Omega), T_w^{1,q}(\partial\Omega)\right]_\beta, \tag{8.1.4}$$

equipped with the norm of the interpolation space.

To ensure the well-definedness of the boundary conditions we need to demand that the force F and the divergence K are contained in some space of distributions on Ω. Since Sobolev embeddings require strong assumptions to the weight function w we assume (8.1.7) below. See Lemma 10.1.4 below for sufficient conditions for (8.1.7).

Theorem 8.1.4. *Let $1 < q < \infty$, $w \in A_q$ and $0 \le \beta \le 1$. Assume that $f \in Y_w^{-2,q}(\Omega)$ and $k \in H_{w,0}^{-1,q}(\Omega)$ allow decompositions into*

$$\begin{aligned} \langle f,\phi\rangle &= \langle F,\phi\rangle - \langle g, N\cdot\nabla\phi\rangle_{\partial\Omega} && \text{for every } \phi \in Y_{w'}^{2,q'}(\Omega) \\ \langle k,\psi\rangle &= \langle K,\psi\rangle - \langle g, N\psi\rangle_{\partial\Omega} && \text{for every } \psi \in W_{w'}^{1,q'}(\Omega) \end{aligned} \tag{8.1.5}$$

with $F \in Y_w^{\beta-2,q}(\Omega)$, $K \in H_{w,0}^{\beta-1,q}(\Omega)$ and $g \in T_w^{\beta,q}(\partial\Omega)$.

Assume in addition that K and g fulfill the compatibility condition

$$\langle K, 1\rangle_\Omega = \langle g, N\rangle_{\partial\Omega}.$$

Then the very weak solution $u \in L_w^q(\Omega)$ with respect to f and k, which exists according to Theorem 6.1.4 is contained in $H_w^{\beta,q}(\Omega)$ and fulfills the estimate

$$\|u\|_{\beta,q,w} \le c\left(\|F\|_{Y_w^{\beta-2,q}(\Omega)} + \|K\|_{H_{w,0}^{\beta-1,q}(\Omega)} + \|g\|_{T_w^{\beta,q}(\partial\Omega)}\right). \tag{8.1.6}$$

Moreover, if we assume in addition that $F \in W_{\tilde w}^{-1,r}(\Omega)$ and $K \in L_{\tilde w}^r(\Omega)$, where r and $\tilde w \in A_r$ are chosen such that

$$W_{\tilde w}^{-1,r}(\Omega) \hookrightarrow Y_w^{\beta-2,q}(\Omega) \quad \text{and} \quad L_{\tilde w}^r(\Omega) \hookrightarrow H_{w,0}^{\beta-1,q}(\Omega). \tag{8.1.7}$$

then $u|_{\partial\Omega}$ is well-defined in the sense of (6.3.6) and one has $u|_{\partial\Omega} = g$.

Proof. Step 1: We consider the operator

$$B : T_w^{0,q}(\partial\Omega) \to L_w^q(\Omega), \quad g \mapsto u,$$

where u is the very weak solution to the Stokes problem with data

$$f = [\phi \mapsto \langle g, N \cdot \nabla\phi\rangle_{\partial\Omega}] \text{ and } k = [\psi \mapsto \langle g, N\psi\rangle_{\partial\Omega}].$$

Obviously, B is linear and continuous, also considered as an operator $B : T_w^{1,q}(\partial\Omega) \to W_w^{1,q}(\Omega)$. This follows from Theorem 8.1.3 in the case $\beta = 1$ since the very weak solution with respect to f and k coincides with the weak solution with 0 force and divergence and boundary condition g. Thus interpolation yields that

$$B : T_w^{\beta,q}(\partial\Omega) \to H_w^{\beta,q}(\Omega)$$

is continuous.

Step 2: Let $U = Bg \in H_w^{\beta,q}(\Omega)$ be given by Step 1. Moreover, let $v \in Y_w^{\beta,q}(\Omega)$ be the very weak solution to the Stokes problem with respect to the data F, K, which exists according to Theorem 8.1.1 and fulfills the estimate

$$\|v\|_{\beta,q,w} \leq c\left(\|F\|_{Y_w^{\beta-2,q}(\Omega)} + \|K\|_{H_{w,0}^{\beta-1,q}(\Omega)}\right).$$

Note that this v fulfills $v|_{\partial\Omega} = 0$ if v is regular enough to ensure that such a trace is well-defined. Now we set $u := U + v$. Then u is a very weak solution with respect to f and k and fulfills the estimate (8.1.6).

Moreover, if $F \in W_{\tilde{w}}^{-1,r}(\Omega)$ and $K \in L_{\tilde{w}}^r(\Omega)$, then by Proposition 6.3.7 we obtain $u|_{\partial\Omega} = g$ in the sense of (6.3.6).

The uniqueness of the solution follows from Theorem 6.1.4. □

Note that the data in Theorem 8.1.4 is in general not regular enough to ensure that the restriction to the boundary is well-defined. Accordingly, the decomposition of the data is in general not unique. This uniqueness is only guaranteed if $F \in W_{\tilde{w}}^{-1,r}(\Omega)$ and $K \in L_{\tilde{w}}^r(\Omega)$.

Remark 8.1.5. An analogous version of Theorem 8.1.4 holds if the decomposition (8.1.5) holds only for f or for k. I.e., f can be decomposed into a more regular force F and the tangential component of g or k can be decomposed into a more regular divergence K and the normal component of the boundary condition g. Naturally then only the tangential or the normal trace are well-defined.

Corollary 8.1.6. *Let Ω be a bounded $C^{1,1}$-domain. Moreover, let $1 < q, r < \infty$, $w \in A_q$, $v \in A_r$ and $0 \leq \beta \leq 2$ be given such that $H_w^{\beta,q}(\Omega) \hookrightarrow L_v^r(\Omega)$. Then*

$$T_w^{\beta,q}(\partial\Omega) \hookrightarrow T_v^{0,r}(\partial\Omega).$$

Proof. Let $g \in T_w^{\beta,q}(\partial\Omega)$. Then the very weak solution $u \in H_w^{\beta,q}(\Omega)$ to

$$\begin{aligned} -\langle u, \Delta\phi\rangle &= \langle g, N \cdot \nabla\phi\rangle_{\partial\Omega} \quad \text{for all } \phi \in Y_{w',\sigma}^{2,q'}(\Omega) \\ -\langle u, \nabla\psi\rangle &= \langle g, N\psi\rangle_{\partial\Omega} \qquad \text{for all } \psi \in W_{w'}^{1,q'}(\Omega) \end{aligned}$$

fulfills $\|u\|_{\beta,q,w} \leq c\|g\|_{T_w^{\beta,q}(\partial\Omega)}$. Moreover, one has $u \in \tilde{W}_{v,v}^{r,r}$ (defined in (6.3.1)) with $\|u\|_{\tilde{W}_{v,v}^{r,r}} = \|u\|_{r,v}$ and $\operatorname{div} u = 0$. Thus the tangential and the normal trace of u are well-defined in the sense of (6.3.6). Since $u|_{\partial\Omega} = g$, we obtain

$$\|g\|_{T_v^{0,r}(\partial\Omega)} \leq c\|u\|_{r,v} \leq c\|u\|_{\beta,q,w} \leq c\|g\|_{T_w^{\beta,q}(\partial\Omega)}.$$

□

The results of this section can be used for the proof of the following Lemma which is needed to estimate the pressure in the instationary case. Since the pressure is well-defined only modulo constants, we consider the space $H^{\beta,q}_w(\Omega)/const.$ If $\beta \geq 0$ this space can be identified with the space of all $u \in H^{\beta,q}_w(\Omega)$ such that $\langle u, 1\rangle_\Omega = 0$. If $\beta < 0$ one has

$$H^{\beta,q}_w(\Omega)/const. \cong \left\{ \phi \in H^{-\beta,q'}_{w',0}(\Omega) \;\middle|\; \int_\Omega \phi = 0 \right\}'.$$

Proof. Take $\zeta \in H^{-\beta,q'}_{w',0}(\Omega)$ with $\int \zeta = 1$. For $u \in \left\{ \phi \in H^{-\beta,q'}_{w',0}(\Omega) \;\middle|\; \int_\Omega \phi = 0 \right\}'$ we set

$$\langle I(u), \phi\rangle_\Omega = \left\langle u, \phi - \zeta \int \phi \right\rangle_\Omega.$$

Then $I(u) + \mathbb{R} \in H^{\beta,q}_w(\Omega)/const.$ More precisely

$$u \mapsto I(u) + \mathbb{R} : \left\{ \phi \in H^{-\beta,q'}_{w',0}(\Omega) \;\middle|\; \int_\Omega \phi = 0 \right\}' \to H^{\beta,q}_w(\Omega)/const.$$

is a continuous isomorphism. Its inverse is

$$u + \mathbb{R} \mapsto u|_{\left\{\phi \in H^{-\beta,q'}_{w',0}(\Omega) \;\mid\; \int_\Omega \phi = 0\right\}}$$

for every $u \in H^{\beta,q}_w(\Omega)$. □

Lemma 8.1.7. *Let $-1 \leq \beta \leq 1$. Let $p \in (C_0^\infty(\Omega))'$ with $\nabla p \in H^{\beta-1,q}_w(\Omega)$. Then $p \in H^{\beta,q}_w(\Omega)$ and there exists a constant $c = c(\Omega, q, w)$ such that*

$$\|p\|_{H^{\beta,q}_w/const.} \leq c\|\nabla p\|_{H^{\beta-1,q}_w}.$$

Proof. Case 1: Let $\beta \leq 0$. By Theorem 4.3.1 for every $\phi \in W^{1,q'}_{w',0}(\Omega)$ with $\int_\Omega \phi = 0$ there exists $\zeta \in W^{2,q'}_{w',0}(\Omega)$ such that $\operatorname{div} \zeta = \phi$ and $\|\zeta\|_{2,q',w'} \leq c\|\phi\|_{1,q',w'}$. The function ζ can be chosen such that the mapping $\phi \mapsto \zeta$ is linear and fulfills the additional estimate $\|\zeta\|_{1,q',w'} \leq c\|\phi\|_{q',w'}$.

For a moment we consider the mapping $\phi \mapsto \zeta$ as a mapping from $L^{q'}_{w'}(\Omega)$ to $H^{1,q'}_{w'}(\mathbb{R}^n)$ and from $H^{1,q'}_{w',0}(\Omega)$ to $H^{2,q'}_{w'}(\mathbb{R}^n)$ assuming that ζ is extended by 0 to a function defined on $\mathbb{R}^n$. Thus by interpolation we obtain for $\gamma \in [0, 1]$

$$\|\zeta\|_{H^{\gamma+1,q'}_{w'}(\mathbb{R}^n)} \leq c\|\phi\|_{H^{\gamma,q'}_{w',0}(\Omega)}.$$

Since for $\phi \in C_0^\infty(\Omega)$ one has $\operatorname{supp} \zeta \subset \Omega$, we have shown

$$\|\zeta\|_{H^{\gamma+1,q'}_{w',0}(\Omega)} \leq c\|\phi\|_{H^{\gamma,q'}_{w',0}(\Omega)}.$$

This implies the estimate

$$|\langle p, \phi\rangle_\Omega| = |\langle p, \operatorname{div} \zeta\rangle_\Omega| \leq \|\nabla p\|_{H^{\beta-1,q}_w}\|\zeta\|_{H^{1-\beta,q'}_{w',0}} \leq c\|\nabla p\|_{H^{\beta-1,q}_w}\|\phi\|_{H^{-\beta,q'}_{w',0}}$$

for every $\phi \in C_0^\infty(\Omega)$. This is the assertion for $\beta \le 0$.
Case 2: Let $\beta > 0$. Then for every $\phi \in H_{w',0}^{-\beta,q'}(\Omega)$ with $\langle \phi, 1\rangle_\Omega = 0$ by Theorem 8.1.1 there exists a very weak solution $\zeta \in Y_{w'}^{1-\beta,q'}(\Omega)$ to the Stokes problem with force 0 and divergence ϕ. In particular,

$$\langle \zeta, \nabla\psi\rangle_\Omega = \langle \phi, \psi\rangle \quad \text{for every} \quad \psi \in W_w^{1,q}(\Omega), \tag{8.1.8}$$

and ζ fulfills the estimate $\|\zeta\|_{Y_{w'}^{1-\beta,q'}} \le c\|\phi\|_{H_{w',0}^{-\beta,q'}}$. Note that in the case $0 \le 1-\beta \le 1$ one has $Y_{w'}^{1-\beta,q'}(\Omega) = H_{w',0}^{1-\beta,q'}(\Omega)$. Thus we obtain if p is sufficiently smooth using (8.1.8)

$$\begin{aligned} |\langle p, \phi\rangle_\Omega| = |\langle \nabla p, \zeta\rangle_\Omega| &\le \|\nabla p\|_{H_w^{\beta-1,q}} \|\zeta\|_{Y_{w'}^{1-\beta,q'}} \\ &\le c\|\nabla p\|_{H_w^{\beta-1,q}} \|\phi\|_{H_{w',0}^{-\beta,q'}}. \end{aligned} \tag{8.1.9}$$

This completes the proof for smooth p since $H_{w',0}^{-\beta,q'}(\Omega) = (H_w^{\beta,q}(\Omega))'$.
For general p we use approximations as shown in the following.
Choose $\Omega' \subset \overline{\Omega'} \subset \Omega$. Moreover, choose a mollifyer ρ_ε such that $\operatorname{supp} \rho_\varepsilon(x - \cdot) \subset \Omega$ for every $x \in \Omega'$ and for every $0 < \varepsilon < \varepsilon_0$. Then one has $p * \rho_\varepsilon \in C^\infty(\overline{\Omega'})$, $p * \rho_\varepsilon \xrightarrow{\varepsilon \to 0} p$ in $C_0^\infty(\Omega)$ and

$$\nabla(p * \rho_\varepsilon) = (\nabla p) * \rho_\varepsilon \xrightarrow{\varepsilon \to 0} \nabla p \;\text{ in }\; H_w^{\beta-1,q}(\Omega).$$

For $\phi \in C_0^\infty(\Omega')$ the application of (8.1.9) in Ω' yields

$$|\langle p * \rho_\varepsilon, \phi\rangle_{\Omega'}| \le c_{\Omega'} \|\nabla(p * \rho_\varepsilon)\|_{H_w^{\beta-1,q}(\Omega')} \|\phi\|_{H_{w',0}^{-\beta,q'}(\Omega')}.$$

For $\varepsilon \to 0$ we obtain $p \in H_w^{\beta,q}(\Omega')$ with $\|p\|_{H_w^{\beta,q}(\Omega')} \le c_{\Omega'}\|\nabla p\|_{H_w^{\beta-1,q}(\Omega')}$. Note that the constant $c_{\Omega'}$ arises from the constant in the a priori estimate for solutions to the Stokes equations.
We have to choose subsets Ω' such that these constants do not explode. To do this in an appropriate way we decompose Ω in strictly star-shaped $C^{1,1}$-domains. More precisely, we find open sets (U_j) such that $\overline{\Omega} \subset \bigcup_{j=1}^N U_j$ and each $\Omega_j := \Omega \cap U_j$ is a strictly starshaped $C^{1,1}$-domain. Assume without loss of generality that Ω_j is starshaped with respect to 0.
The following scaling argument uses that each Ω_j is starshaped with respect to every point of an open ball centered at 0. This can be guaranteed by a temporary translation of the system of coordinates. By $\lambda\Omega_j$, $0 < \lambda < 1$, we denote the subset of Ω_j which is obtained by scaling Ω_j with respect to the origin of the shifted system of coordinates which depends on j.
For $\lambda \in (\frac{1}{2}, 1)$ let $\zeta \in H_{w'}^{1-\beta,q'}(\lambda\Omega_j)$ be a solution to

$$-\Delta\zeta + \nabla p = 0, \quad \operatorname{div}\zeta = \phi, \quad \zeta|_{\partial(\lambda\Omega_j)} = 0,$$

for $\phi \in C_0^\infty(\lambda\Omega_j)$. Then

$$\frac{1}{\lambda}\zeta(\lambda x) \in H_{w'(\lambda\cdot)}^{1-\beta,q'}(\Omega_j)$$

solves the same equation in Ω_j with ϕ replaced by $\phi(\lambda x)$. In particular, we obtain

$$\|\zeta\|_{H_{w'}^{1-\beta,q'}(\lambda\Omega_j)} \le c\left\|\frac{1}{\lambda}\zeta(\lambda x)\right\|_{H_{w'(\lambda\cdot)}^{1-\beta,q'}(\Omega_j)} \le c\|\phi(\lambda\cdot)\|_{H_{w'(\lambda\cdot)}^{-\beta,q'}(\Omega_j)} \le c\|\phi\|_{H_{w'}^{-\beta,q'}(\lambda\Omega_j)},$$

where the constant c depends on the domain Ω_j, contains some multiples of $\lambda \in (\frac{1}{2}, 1)$ and depends A_q-consistently on $w'(\lambda\cdot)$. Since $A_q(w'(\lambda\cdot))$ is bounded for $\lambda \in (\frac{1}{2}, 1)$, the constant c can be chosen independent of λ.

Thus one obtains $\|p\|_{H_w^{\beta,q}(\lambda\Omega_j)} \le c\|\nabla p\|_{H_w^{\beta-1,q}(\lambda\Omega_j)}$ with a constant c independent of λ.

In particular, if $\beta = 0$ then the proof is complete and therefore we may assume from now on that $\beta > 0$.

Choose an increasing sequence (λ_k) converging to 1. Since $p|_{\lambda_k\Omega_j} \in H_w^{\beta,q}(\lambda_k\Omega_j)$, there exists $P_k \in H_w^{\beta,q}(\mathbb{R}^n)$ with $P_k|_{\lambda_k\Omega_j} = p|_{\lambda_k\Omega_j}$ and

$$\|P_k\|_{H_w^{\beta,q}(\mathbb{R}^n)} \le 2\|p\|_{H_w^{\beta,q}(\lambda_k\Omega_j)} \le 2c\|\nabla p\|_{H_w^{\beta-1,q}(\Omega)}.$$

Thus we may assume without loss of generality that $P_k \rightharpoonup P$ in $H_w^{\beta,q}(\mathbb{R}^n)$ and $P_k|_{\Omega_j} \rightharpoonup P|_{\Omega_j}$ in $H_w^{\beta,q}(\Omega_j)$.

By (3.2.2) there exists $r > 1$ such that

$$H_w^{\beta,q}(\Omega_j) \hookrightarrow H^{\beta,r}(\Omega_j) \hookrightarrow L^r(\Omega_j),$$

where the last embedding is compact, see e.g. [13], 3.3. This implies $P_k|_{\Omega_j} \to P|_{\Omega_j}$ in $L^r(\Omega_j)$. Moreover, it is obvious that P_k converges in the pointwise sense to p and we obtain $P|_{\Omega_j} = p$ and this shows $p \in H_w^{\beta,q}(\Omega_j)$.

Now let P_j be the extension of $p|_{\Omega_j}$ constructed above. Set $P := \sum_{j=1}^n \phi_j P_j$, where (ϕ_j) is a partition of unity subordinate to the covering (U_j). Then one has $P|_\Omega = p$ and

$$\|p\|_{H_w^{\beta,q}(\Omega)} \le \|P\|_{H_w^{\beta,q}(\mathbb{R}^n)} \le c\sum_{j=1}^N \|P_j\|_{H_w^{\beta,q}(\mathbb{R}^n)} \le 2Nc\|\nabla p\|_{H_w^{\beta-1,q}(\Omega)}.$$

□

8.2 The Generalized Stokes Operator

For this section we always assume that $q \in (1, \infty)$, $w \in A_q$ and $\beta \in [0, 2]$.

As a divergence free version of the space $Y_w^{-2,q}(\Omega)$ we define

$$Y_{w,\sigma}^{-2,q}(\Omega) := \left(Y_{w',\sigma}^{2,q'}(\Omega)\right)'.$$

By the Hahn-Banach theorem the space $Y_{w,\sigma}^{-2,q}(\Omega)$ is the restriction of all elements of $Y_w^{-2,q}(\Omega)$ to $Y_{w',\sigma}^{2,q'}(\Omega)$. We define the divergence free version of $Y_w^{\beta,q}(\Omega)$ by

$$Y_{w,\sigma}^{\beta,q}(\Omega) := \left\{u \in Y_w^{\beta,q}(\Omega) \mid \langle u, \nabla\phi\rangle = 0 \text{ for every } \phi \in C^\infty(\overline{\Omega})\right\}.$$

For $\beta = 2$ this definition is consistent with the one given in (6.1.5) and by Theorem 7.4.2 and (4.4.1) one has

$$Y_{w,\sigma}^{1,q}(\Omega) = W_{w,0,\sigma}^{1,q}(\Omega) \text{ and } Y_{w,\sigma}^{0,q}(\Omega) = L_{w,\sigma}^q(\Omega).$$

Lemma 8.2.1. *Let $u \in Y_w^{\beta,q}(\Omega)$. Then the mapping $\phi \mapsto (u, \Delta\phi)$ for $\phi \in Y_{w'}^{2,q'}(\Omega)$ can be extended to an element of $Y_w^{\beta-2,q}(\Omega)$.*

Proof. Case 1 $(0 \le \beta \le 1)$:
In this case, $Y_{w'}^{-\beta,q'}(\Omega)$ consists of distributions on Ω. We show that

$$\Delta : Y_{w'}^{2-\beta,q'}(\Omega) \to Y_{w'}^{-\beta,q'}(\Omega) = H_{w'}^{-\beta,q'}(\Omega)$$

is continuous.

Since $1 \le 2-\beta \le 2$, the space $Y_{w'}^{2-\beta,q'}(\Omega)$ is equipped with the norm in $H_{w'}^{2-\beta,q'}(\Omega)$. Thus for every $\phi \in Y_{w'}^{2,q'}(\Omega) \hookrightarrow Y_{w'}^{2-\beta,q'}(\Omega)$ there exists $\psi \in H_{w'}^{2-\beta,q'}(\mathbb{R}^n)$ with $\psi|_\Omega = \phi$ and $\|\psi\|_{2-\beta,q,w,\mathbb{R}^n} \le 2\|\phi\|_{2-\beta,q,w,\Omega}$. Then $\Delta\psi \in H_{w'}^{-\beta,q'}(\mathbb{R}^n)$ is an extension of $\Delta\phi$, and we obtain by the definition of the norms

$$\begin{aligned} |(u,\Delta\phi)| &\le \|u\|_{Y_w^{\beta,q}(\Omega)}\|\Delta\phi\|_{Y_{w'}^{-\beta,q'}(\Omega)} \le \|u\|_{Y_w^{\beta,q}(\Omega)}\|\Delta\psi\|_{H_{w'}^{-\beta,q'}(\mathbb{R}^n)} \\ &\le c\|u\|_{Y_w^{\beta,q}(\Omega)}\|\psi\|_{H_{w'}^{2-\beta,q'}(\mathbb{R}^n)} \le c\|u\|_{Y_w^{\beta,q}(\Omega)}\|\phi\|_{Y_{w'}^{2-\beta,q'}(\Omega)}. \end{aligned}$$

This shows that $\phi \mapsto (u,\Delta\phi)$ extends to an element of $(Y_{w'}^{2-\beta,q'}(\Omega))' = Y_w^{\beta-2,q}(\Omega)$.
Case 2 $(1 \le \beta \le 2)$:
In this case one has $u|_{\partial\Omega} = \phi|_{\partial\Omega} = 0$ for $u \in Y_w^{\beta,q}(\Omega)$ and $\phi \in Y_{w'}^{2,q'}(\Omega)$. Thus integration by parts yields

$$(u,\Delta\phi)_\Omega = -(\nabla u,\nabla\phi)_\Omega.$$

Now $\nabla u \in H_w^{\beta-1,q}(\Omega)$ and the distributional derivative

$$\operatorname{div} : H_w^{\beta-1,q}(\Omega) \to H_w^{\beta-2,q}(\Omega) = Y_w^{\beta-2,q}(\Omega)$$

is continuous which means

$$\langle \Delta u,\zeta\rangle_\Omega = \langle \operatorname{div}\nabla u,\zeta\rangle_\Omega = \langle \nabla u,\nabla\zeta\rangle_\Omega \quad \text{for} \quad \zeta \in C_0^\infty(\Omega). \tag{8.2.1}$$

Now $\phi \in Y_{w'}^{2,q'}(\Omega)$ can be approximated in $W_{w',0}^{1,q'}(\Omega)$ by $C_0^\infty(\Omega)$-functions and it follows that (8.2.1) holds for ζ replaced by $\phi \in Y_{w'}^{2,q'}(\Omega)$. Combining the above yields

$$|(u,\Delta\phi)_\Omega| = |\langle \Delta u,\phi\rangle_\Omega| \le \|\Delta u\|_{Y_w^{\beta-2,q}(\Omega)}\|\phi\|_{Y_{w'}^{2-\beta,q'}(\Omega)} \le c\|u\|_{Y_w^{\beta,q}(\Omega)}\|\phi\|_{Y_{w'}^{2-\beta,q'}(\Omega)},$$

where the last inequality holds by the same arguments as in Case 1. □

Corollary 8.2.2. *If* $1 < q < \infty$, $w \in A_q$, $0 \le \beta \le 2$ *and if* $\Omega \subset \mathbb{R}^n$ *is a bounded* $C^{1,1}$*-domain, then*

$$\left[L_{w,\sigma}^q(\Omega), Y_{w,\sigma}^{2,q}(\Omega)\right]_\theta = Y_{w,\sigma}^{\beta,q}(\Omega)$$

and

$$\left[L_{w,\sigma}^q(\Omega), Y_{w,\sigma}^{-2,q}(\Omega)\right]_\theta = Y_{w,\sigma}^{-\beta,q}(\Omega),$$

where $\theta = \frac{\beta}{2}$ *with equivalent norms.*

Proof. "$\subset$" is proved in the same way as in Step 1 of the proof of Theorem 7.4.1.

It remains to prove "$\supset$". From Theorem 8.1.1 we obtain that the operator

$$\mathcal{S} : Y_w^{\beta-2,q}(\Omega) \ni f \mapsto u \in Y_w^{\beta,q}(\Omega),$$

where u is the solution of

$$\begin{aligned} \langle f, \varphi\rangle &= -\langle u, \Delta\varphi\rangle && \text{for all } \varphi \in Y^{2;q'}_{w',\sigma}(\Omega) \text{ and} \\ 0 &= -\langle u, \nabla\psi\rangle && \text{for all } \psi \in W^{1;q}_{w'}(\Omega), \end{aligned} \tag{8.2.2}$$

is continuous. We show that the image space equals $Y^{\beta;q}_{w,\sigma}(\Omega)$. By Lemma 8.2.1 one has

$$Au = f = [\phi \mapsto \langle u, \Delta\phi\rangle] \in Y^{\beta-2,q}_{w}(\Omega)$$

for every $u \in Y^{\beta;q}_{w,\sigma}(\Omega)$. Since this u is the solution of (8.2.2) with respect to f, one has $u = \mathcal{S}f$.

Moreover, using $Y^{\beta-2,q}_{w}(\Omega) = [Y^{-2,q}_{w}(\Omega), L^q_w(\Omega)]_\theta$ interpolation shows that

$$\mathcal{S} : Y^{\beta-2,q}_{w}(\Omega) \to \left[L^q_{w,\sigma}(\Omega), Y^{2;q}_{w,\sigma}(\Omega)\right]_\theta$$

is continuous. Thus we find for $x \in Y^{\beta;q}_{w,\sigma}(\Omega)$

$$\begin{aligned} \|x\|_{[L^q_{w,\sigma}(\Omega), Y^{2;q}_{w,\sigma}(\Omega)]_\theta} &= \|\mathcal{S}Ax\|_{[L^q_{w,\sigma}(\Omega), Y^{2;q}_{w,\sigma}(\Omega)]_\theta} \le c\|Ax\|_{Y^{\beta-2,q}_{w}(\Omega)} \\ &\le c\|x\|_{Y^{\beta,q}_{w}(\Omega)} = c\|x\|_{Y^{\beta,q}_{w,\sigma}(\Omega)}. \end{aligned}$$

This implies $Y^{\beta;q}_{w,\sigma}(\Omega) \hookrightarrow \left[L^q_{w,\sigma}(\Omega), Y^{2;q}_{w,\sigma}(\Omega)\right]_\theta$.

The second assertion follows when considering the dual spaces in the first. □

As in the classical unweighted case one defines the Stokes operator

$$\mathcal{A} = \mathcal{A}_{0,q,w} : L^q_{w,\sigma}(\Omega) \supset \mathcal{D}(\mathcal{A}) \to L^q_{w,\sigma}(\Omega), \quad u \mapsto -P_{q,w}\Delta,$$

where $P_{q,w}$ is the Helmholtz projection defined in Section 4.4. Its domain $\mathcal{D}(\mathcal{A}) = Y^{2;q}_{w,\sigma}(\Omega)$. In the weighted context it has been introduced and discussed in [27] and [26].

In the following, we find an analogue to the Stokes operator which is adequate in the context of very weak solutions in the Bessel potential spaces $H^{\beta,q}_{w}(\Omega)$.

Theorem 8.2.3. *For every $0 \le \beta \le 2$ the Stokes operator $\mathcal{A}$ has an extension to an element of $\mathcal{L}(Y^{\beta;q}_{w,\sigma}(\Omega), Y^{\beta-2,q}_{w,\sigma}(\Omega))$ with the following properties.*

1. *It describes a closed and densely defined linear operator in $Y^{\beta-2,q}_{w,\sigma}(\Omega)$ again denoted by $\mathcal{A}$. For $u \in Y^{\beta;q}_{w,\sigma}(\Omega)$ one has*

$$\mathcal{A}u = [Y^{2-\beta,q'}_{w',\sigma}(\Omega) \ni \phi \mapsto -\langle u, \Delta\phi\rangle_\Omega].$$

2. *The resolvent set $\rho(-\mathcal{A})$ contains a sector $\Sigma_\varepsilon \cup \{0\}$, $\varepsilon \in (0, \frac{\pi}{2})$, and for $\lambda \in \Sigma_\varepsilon \cup \{0\}$ the operator $\lambda + \mathcal{A}$ is an isomorphism in $\mathcal{L}(Y^{\beta;q}_{w,\sigma}(\Omega), Y^{\beta-2,q}_{w,\sigma}(\Omega))$. The norm of the inverse $\|(\lambda + \mathcal{A})^{-1}\|_{\mathcal{L}(Y^{\beta-2,q}_{w,\sigma}, Y^{\beta,q}_{w,\sigma})}$ is independent of $\lambda \in \Sigma_\delta$ for every $0 < \delta < \varepsilon$.*

3. *For every $0 < \delta < \varepsilon$ there exists a constant M_δ such that*

$$\|\lambda(\mathcal{A} + \lambda)^{-1}\|_{\mathcal{L}(Y^{\beta-2,q}_{w,\sigma}(\Omega))} \le M_\delta \quad \textit{for all} \quad \lambda \in \Sigma_\delta. \tag{8.2.3}$$

For $-2 \le \mu \le 0$ let $\mathcal{A}_{\mu,q,w}$ be the extension of the Stokes operator whose existence has been stated in Theorem 8.2.3. Then we call

$$\mathcal{A}_{\mu,q,w} : \mathcal{D}(\mathcal{A}_{\mu,q,w}) := Y^{\mu+2,q}_{w,\sigma}(\Omega) \subset Y^{\mu,q}_{w,\sigma}(\Omega) \to Y^{\mu,q}_{w,\sigma}(\Omega)$$

the generalized Stokes operator in $Y^{\mu,q}_{w,\sigma}(\Omega)$. If no confusion can occur, we omit the indices and write $\mathcal{A}$ instead of $\mathcal{A}_{\mu,q,w}$.

Proof. For $\beta = 2$ one has $Y^{\beta,q}_{w,\sigma}(\Omega) = Y^{2,q}_{w,\sigma}(\Omega) = \mathcal{D}(\mathcal{A})$, the domain of the classical Stokes operator in $L^q_{w,\sigma}(\Omega)$. Hence, in this case the assertion of this theorem is shown in [25, 7.3], where the Stokes operator in $L^q_{w,\sigma}(\Omega)$ is introduced.

Our aim is to show the assertion for $\beta = 0$ and to apply complex interpolation to obtain the results for arbitrary $0 \le \beta \le 2$.

Step 1: We consider $\lambda + \mathcal{A}_{0,q',w'}$, where $\mathcal{A}_{0,q',w'}$ is the Stokes operator in $L^{q'}_{w',\sigma}(\Omega)$, as a continuous linear operator

$$\lambda + \mathcal{A}_{0,q',w'} : Y^{2,q'}_{w',\sigma}(\Omega) \to L^{q'}_{w',\sigma}(\Omega).$$

Let $\mathcal{A}_{-2,q,w} = \mathcal{A}^*_{0,q',w'}$ be the associated dual operator. Then one has for $u \in Y^{2,q}_{w,\sigma}(\Omega)$ and $\phi \in Y^{2,q'}_{w',\sigma}(\Omega)$

$$\begin{aligned}\langle(\lambda + \mathcal{A}_{-2,q,w})u, \phi\rangle &= \langle u, (\lambda + \mathcal{A}_{0,q',w'})\phi\rangle = \langle u, \lambda\phi - \Delta\phi\rangle \\ &= \langle(\lambda - P_{q,w}\Delta)u, \phi\rangle = \langle(\lambda + \mathcal{A}_{0,q',w'})u, \phi\rangle.\end{aligned}$$

Thus we obtain using the properties of the dual operator [10]

- $(\lambda + \mathcal{A}_{-2,q,w})|_{Y^{2,q}_{w,\sigma}} = (\lambda + \mathcal{A}_{0,q,w})|_{Y^{2,q}_{w,\sigma}}$.

- For $\lambda \in \Sigma_\varepsilon \cup \{0\}$ one has

$$\lambda + \mathcal{A}_{-2,q,w} = (\lambda + \mathcal{A}_{0,q',w'})^*,$$

  which implies $\|\lambda + \mathcal{A}_{-2,q,w}\|_{\mathcal{L}(L^q_{w,\sigma}, Y^{-2,q}_{w,\sigma})} = \|\lambda + \mathcal{A}_{0,q',w'}\|_{\mathcal{L}(Y^{2,q'}_{w',\sigma}, L^{q'}_{w',\sigma})}$.

- $\Sigma_\varepsilon \cup \{0\}$ is contained in the resolvent set of $\mathcal{A}_{-2,q,w}$ and there exists $M_\delta > 0$ such that for all $\lambda \in M_\delta$, $0 < \delta < \varepsilon$,

$$\|(\lambda + \mathcal{A}_{-2,q,w})^{-1}\|_{\mathcal{L}(Y^{-2,q}_{w,\sigma}, L^q_{w,\sigma})} = \|(\lambda + \mathcal{A}_{0,q',w'})^{-1}\|_{\mathcal{L}(L^{q'}_{w',\sigma}, Y^{2,q'}_{w',\sigma})} \le M_\delta.$$

  This implies

$$\begin{aligned}&\|\lambda(\lambda + \mathcal{A}_{-2,q,w})^{-1}f\|_{Y^{-2,q}_{w,\sigma}} + \|(\lambda + \mathcal{A}_{-2,q,w})^{-1}f\|_{q,w} \\ &\qquad \le \|f\|_{Y^{-2,q}_{w,\sigma}} + \|\mathcal{A}_{-2,q,w}(\lambda + \mathcal{A}_{-2,q,w})^{-1}f\|_{Y^{-2,q}_{w,\sigma}} + \|(\lambda + \mathcal{A}_{-2,q,w})^{-1}f\|_{q,w} \\ &\qquad \le M_\delta\|f\|_{Y^{-2,q}_{w,\sigma}}.\end{aligned}$$

Since the resolvent set is nonempty, we know that the operator $\mathcal{A}_{-2,q,w}$ is closed in $Y^{-2,q}_{w,\sigma}(\Omega)$. Using the Hahn-Banach theorem one shows that $L^q_{w,\sigma}(\Omega)$, which is equal to the domain of $\mathcal{A}_{-2,q,w}$ in $Y^{-2,q}_{w,\sigma}(\Omega)$, is dense in $Y^{-2,q}_{w,\sigma}(\Omega)$.
Step 2: Combining Corollary 8.2.2 and the assertion for $\beta = 0$ and $\beta = 2$ we obtain by complex interpolation that

$$\mathcal{A} : Y^{\beta,q}_{w,\sigma}(\Omega) \to Y^{\beta-2,q}_{w,\sigma}(\Omega) \quad \text{and} \quad (\lambda - \mathcal{A})^{-1} : Y^{\beta-2,q}_{w,\sigma}(\Omega) \to Y^{\beta,q}_{w,\sigma}(\Omega), \quad \lambda \in \Sigma_\delta \cup \{0\}$$

are continuous operators. Theorem 2.3.1 and (8.2.3) for $\beta = 0$ and $\beta = 2$ imply in addition

$$\begin{aligned} \|(\lambda + \mathcal{A})^{-1}\|_{\mathcal{L}(Y^{\beta-2,q}_{w,\sigma}(\Omega))} &\leq \|(\lambda + \mathcal{A})^{-1}\|^{\frac{\beta}{2}}_{\mathcal{L}(L^q_{w,\sigma})} \|(\lambda + \mathcal{A})^{-1}\|^{1-\frac{\beta}{2}}_{\mathcal{L}(Y^{-2,q}_{w,\sigma})} \\ &\leq M_\delta |\lambda|^{-\frac{\beta}{2}} |\lambda|^{-1+\frac{\beta}{2}} = M_\delta |\lambda|^{-1} \end{aligned}$$

for every $\lambda \in \Sigma_\delta$. This completes the proof. □

Recall that for $\varepsilon \in (0, \frac{\pi}{2})$ one defines

$$\Delta_\varepsilon := \{\lambda \in \mathbb{C} \mid \lambda \neq 0, \ |\arg \lambda| < \varepsilon\}.$$

Using the notation given in Section 2.2 one has the following corollary.

Corollary 8.2.4. *The negative generalized Stokes operator $-\mathcal{A}$ in $Y^{\beta-2,q}_{w,\sigma}(\Omega)$ is the generator of a bounded analytic semigroup $\{T(t)\}_{t\in\Delta_\varepsilon}$ for every $\varepsilon \in (0, \frac{\pi}{2})$.*

Proof. This follows immediately when combining Theorem 8.2.3 with Theorem 2.2.2. □

Lemma 8.2.5.

1. *The space of functions $\mathcal{D}(\mathcal{A}_{0,r,1}) \cap Y^{\beta,q}_{w,\sigma}(\Omega)$ is dense in $Y^{\beta,q}_{w,\sigma}(\Omega)$ for every $1 < r < \infty$, where $\mathcal{D}(\mathcal{A}_{0,r,1})$ is the domain of the classical Stokes operator $A_{0,r,1}$ in $L^r(\Omega)$.*

2. *For every $u \in Y^{\beta,q}_{w,\sigma}(\Omega)$ and $v \in Y^{2-\beta,q'}_{w',\sigma}(\Omega)$ one has*

$$\langle \mathcal{A}_{\beta-2,q,w} u, v\rangle = \langle u, \mathcal{A}_{-\beta,q',w'} v\rangle.$$

Proof. 1. We begin to prove the assertion for $\beta = 2$. Recall that $\mathcal{A} : Y^{2,q}_{w,\sigma}(\Omega) \to L^q_{w,\sigma}(\Omega)$ is a topological isomorphism.

Take $f \in C^\infty_{0,\sigma}(\Omega)$ and let $u \in Y^{2,q}_{w,\sigma}(\Omega)$ such that $\mathcal{A}u = f$. This means that u solves $-\Delta u + \nabla p = f$ for some distribution p. By the existence and uniqueness of strong solutions in unweighted spaces we find $u \in \mathcal{D}(\mathcal{A}_{0,r,1})$ for all $1 < r < \infty$.

By Section 4.4 the space $C^\infty_{0,\sigma}(\Omega)$ is dense in $L^q_{w,\sigma}(\Omega)$. Thus

$$\mathcal{D}(\mathcal{A}_{0,r,1}) \cap Y^{2,q}_{w,\sigma}(\Omega) \supset \mathcal{A}^{-1}\left(C^\infty_{0,\sigma}(\Omega)\right)$$

is dense in $Y^{2,q}_{w,\sigma}(\Omega)$.

Theorem 2.3.1.1 combined with Corollary 8.2.2 yields the assertion for arbitrary $0 < \beta \leq 2$.

If $\beta = 0$ then $Y^{\beta,q}_{w,\sigma}(\Omega) = L^q_{w,\sigma}(\Omega)$. Thus the assertion follows from the fact that $C^\infty_{0,\sigma}(\Omega)$ is dense in $L^q_{w,\sigma}$.
2. Choose two sequences

$$(u_n) \subset \mathcal{D}(\mathcal{A}_{0,2,1}) \cap Y^{\beta,q}_{w,\sigma}(\Omega) \quad \text{and} \quad (v_n) \subset \mathcal{D}(\mathcal{A}_{0,2,1}) \cap Y^{2-\beta,q'}_{w',\sigma}(\Omega)$$

such that $u_n \to u$ and $v_n \to v$, where $\mathcal{D}(\mathcal{A}_{0,2,1})$ is the domain of the classical Stokes operator in $L^2(\Omega)$.

From the unweighted theory [47] we know that $A_{0,2,1}$ is self-adjoint. Moreover by [25, Satz 7.6] one has $\mathcal{A}_{0,2,1}u_n = \mathcal{A}_{\beta-2,q,w}u_n$ and $\mathcal{A}_{-\beta,q',w'}v_n = \mathcal{A}_{0,2,1}v_n$. Thus

$$\langle \mathcal{A}_{\beta-2,q,w}u, v\rangle \xleftarrow{n\to\infty} (\mathcal{A}_{0,2,1}u_n, v_n) = (u_n, \mathcal{A}_{0,2,1}v_n) \xrightarrow{n\to\infty} \langle u, \mathcal{A}_{-\beta,q',w'}v\rangle.$$

□

Lemma 8.2.6. *Let $1 < \rho, q < \infty$, $w \in A_q \cap A_\rho$ and $0 \le \beta \le \gamma \le 2$ such that*

$$Y^{\gamma,\rho}_w(\Omega) \hookrightarrow Y^{\beta,q}_w(\Omega).$$

Moreover, let $\mathcal{A}_{\beta-2,q,w}$ and $\mathcal{A}_{\gamma-2,\rho,w}$ be the generalized Stokes operators in $Y^{\beta-2,q}_{w,\sigma}(\Omega)$ and $Y^{\gamma-2,\rho}_{w,\sigma}(\Omega)$, respectively. Then it holds

1. *$P_{q,w}u = P_{\rho,w}u$ for every $u \in L^q_w(\Omega) \cap L^\rho_w(\Omega)$.*
2. *$\mathcal{A}_{\gamma-2,\rho,w}|_{Y^{\gamma,\rho}_{w,\sigma}(\Omega)} = \mathcal{A}_{\beta-2,q,w}|_{Y^{\gamma,\rho}_{w,\sigma}(\Omega)}$.*
3. *For $\lambda \in \Sigma_\delta \cup \{0\}$, $\delta \in (0, \frac{\pi}{2})$, one has*
$$(\lambda + \mathcal{A}_{\gamma-2,\rho,w})^{-1} = (\lambda + \mathcal{A}_{\beta-2,q,w})^{-1}|_{Y^{\gamma-2,\rho}_w(\Omega)}.$$
4. *$e^{-t\mathcal{A}_{\gamma-2,\rho,w}} = e^{-t\mathcal{A}_{\beta-2,q,w}}|_{Y^{\gamma-2,\rho}_{w,\sigma}(\Omega)}$.*

Proof. 1. Let $u \in L^q_w(\Omega) \cap L^\rho_w(\Omega)$. Then $P_{q,w}u = u - \nabla p$, where p is the solution to the weak Neumann problem

$$\langle \nabla p, \nabla \phi\rangle = \langle u, \nabla\phi\rangle \quad \text{for all} \quad \phi \in W^{1,q'}_{w'}(\Omega). \tag{8.2.4}$$

Choose $r > 1$ such that $L^q_w(\Omega) \hookrightarrow L^r(\Omega)$ and $L^\rho_w(\Omega) \hookrightarrow L^r(\Omega)$. Then, since $W^{1,r'}(\Omega) \hookrightarrow W^{1,q'}_{w'}(\Omega)$, we can replace $W^{1,q'}_{w'}(\Omega)$ in (8.2.4) by $W^{1,r'}(\Omega)$. Since the solution to the weak Neumann problem is unique in $W^{1,r}(\Omega)$, it has to coincide with the one in $W^{1,q}_w(\Omega)$, which is given by p and the one in in $W^{1,\rho}_w(\Omega)$. Thus the solution to the Neumann problem does not change with q and ρ and this shows $P_{q,w}u = P_{\rho,w}u$.

2. Let $x \in Y^{\gamma,\rho}_{w,\sigma}(\Omega) = \mathcal{D}(\mathcal{A}_{\gamma-2,\rho,w})$. Then there exists a sequence $(y_n) \subset C^\infty_{0,\sigma}(\Omega) \subset L^\rho_{w,\sigma}(\Omega)$ such that $y_n \to A_{\beta-2,\rho,w}x$ in $Y^{\gamma-2,\rho}_w(\Omega)$.

We show that $x_n := \mathcal{A}^{-1}_{\beta-2,q,w}y_n \in Y^{2,q}_{w,\sigma}(\Omega) \cap Y^{2,\rho}_{w,\sigma}(\Omega)$: Let $r > 1$ be chosen as above. Then the unique strong solution u to the Stokes problem

$$-\Delta u + \nabla p_n = y_n, \quad \operatorname{div} u = 0, \quad u|_{\partial\Omega} = 0$$

is contained in $Y^{2,r}_\sigma(\Omega)$ and by the uniqueness of strong solutions to the Stokes equations this solution coincides with $x_n = \mathcal{A}^{-1}_{\beta-2,q,w} y_n \in Y^{2,q}_w(\Omega)$.

The same arguments show that $x_n = \mathcal{A}^{-1}_{\gamma-2,\rho,w} y_n$, since it is given by the strong solution u to the above Stokes problem. Moreover, by the continuity of $\mathcal{A}^{-1}_{\gamma-2,\rho,w}$ one has $x_n \to x$ in $Y^{\gamma,\rho}_{w,\sigma}(\Omega)$, and by the embedding $Y^{\gamma,\rho}_w(\Omega) \hookrightarrow Y^{\beta,q}_w(\Omega)$ the convergence also holds in $Y^{\beta,q}_w(\Omega)$.

Furthermore, by 1. and the definition of $\mathcal{A}_{\beta-2,q,w}$ in Theorem 8.2.3 one has

$$\mathcal{A}_{\beta-2,q,w} x_n = -P_{q,w}\Delta x_n = -P_{\rho,w}\Delta x_n = \mathcal{A}_{\gamma-2,\rho,w} x_n.$$

Thus one has

$$\mathcal{A}_{\beta-2,q,w} x \xleftarrow{Y^{\beta-2,q}_{w,\sigma}(\Omega)} \mathcal{A}_{\beta-2,q,w} x_n = \mathcal{A}_{\gamma-2,\rho,w} x_n \xrightarrow{Y^{\gamma-2,\rho}_{w,\sigma}(\Omega)} \mathcal{A}_{\gamma-2,\rho,w} x.$$

This shows $\mathcal{A}_{\beta-2,q,w} x = \mathcal{A}_{\gamma-2,\rho,w} x$.

3. This follows from 2. since $\lambda + \mathcal{A}_{\beta-2,q,w} \in \mathcal{L}(Y^{\beta,q}_{w,\sigma}(\Omega), Y^{\beta-2,q}_{w,\sigma}(\Omega))$ and $\lambda + \mathcal{A}_{\gamma-2,\rho,w} \in \mathcal{L}(Y^{\gamma,\rho}_{w,\sigma}(\Omega), Y^{\gamma-2,\rho}_{w,\sigma}(\Omega))$ are isomorphisms.

4. Since the resolvent set of $\mathcal{A}_{\gamma-2,\rho,w}$ contains the same sector $\Sigma_\delta \cup \{0\}$ as the resolvent set of $\mathcal{A}_{\beta,q,w}$, the explicit formula for the semigroup in Section 2.2 yields for $x \in Y^{\gamma-2,\rho}_{w,\sigma}(\Omega)$

$$\begin{aligned} e^{-t\mathcal{A}_{\gamma-2,\rho,w}} x &= \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}(\lambda + \mathcal{A}_{\gamma-2,\rho,w})^{-1} x\, d\lambda \\ &= \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}(\lambda + \mathcal{A}_{\beta-2,q,w})^{-1} x\, d\lambda = e^{-t\mathcal{A}_{\beta-2,q,w}} x, \end{aligned}$$

where $\Gamma \subset \Sigma_\varepsilon$ is a curve from $\infty e^{-i\sigma}$ to $\infty e^{i\sigma}$ for some $\frac{\pi}{2} < \sigma < \frac{\pi}{2} + \varepsilon$.

$\square$

9 Instationary Stokes Equations

We consider the instationary Stokes equations with fully inhomogeneous data on some time interval $[0, T)$, $0 < T \leq \infty$, and a bounded domain $\Omega \subset \mathbb{R}^n$ of class $C^{1,1}$,

$$\begin{aligned} \partial_t u - \Delta u + \nabla p &= F && \text{in } (0,T) \times \Omega, \\ \operatorname{div} u &= K && \text{in } (0,T) \times \Omega, \\ u &= g && \text{on } (0,T) \times \partial\Omega, \\ u(0) &= u_0 && \text{in } \Omega. \end{aligned} \tag{9.0.1}$$

As in the stationary case we multiply the first equation of the Stokes system by a test function and apply formal integration by parts. Then we may treat force and tangential component of the boundary condition as one object. The same is done with the divergence and the normal part of the boundary condition.

With this method the inhomogeneous divergence and boundary condition does not cause any difficulties in addition to the problem $u_t + \mathcal{A}u = f$ as long as we are in the context of lowest regularity, i.e., of solutions $u \in L^r(0,T; L^q_w(\Omega))$.

However, when turning to the nonlinear case we need higher regularity of the solution to estimate the nonlinear term. Thus in Section 9.4 and Section 9.5 we are looking for solutions in weighted Bessel potential spaces. In this case, the inhomogeneous boundary condition and divergence complicates the situation strongly. In particular, one needs some additional time-regularity depending on the order of the Bessel potential space we are working in.

9.1 Very Weak Solutions

As a first step we choose the data and the notion of very weak solutions so general that every $u \in L^r(0,T; L^q_w(\Omega))$ is a very weak solution for appropriate data. However, similarly to the stationary case, a good definition of boundary conditions and initial values requires higher regularity. Thus in a second step we restrict ourselves to smaller function spaces in which we obtain different estimates.

We define some function spaces appropriate to the instationary and very weak context. First, for $T < \infty$ and $1 < r, q < \infty$ we set

$$X^{r',q'}_{w'}(0,T) = \left\{ \phi \in L^{r'}(0,T; Y^{2,q'}_{w'}(\Omega)) \cap W^{1,r'}(0,T; L^{q'}_{w'}(\Omega)) \mid \phi(T) = 0 \right\}$$

and for $T = \infty$

$$\begin{aligned} X^{r',q'}_{w'}(0,\infty) \;=\; & \Big\{ \phi \in L^{r'}(0,\infty; Y^{2,q'}_{w'}(\Omega)) \cap W^{1,r'}(0,\infty; L^{q'}_{w'}(\Omega)) \;\Big| \\ & \quad \operatorname{supp} \phi \text{ compact in } \overline{\Omega} \times [0,\infty) \Big\}. \end{aligned}$$

Both spaces are equipped with the norm

$$\|\phi\|_{X^{r',q'}_{w'}} := \|\phi\|_{L^{r'}(W^{2,q'}_{w'})} + \|\phi_t\|_{L^{r'}(L^{q'}_{w'})}.$$

If there is no danger of confusion, we omit the $(0,T)$ and write $X^{r',q'}_{w'}$. We choose the data

$$f \in \left(X^{r',q'}_{w'}(0,T)\right)' \quad \text{and} \quad k \in L^r(0,T;W^{-1,q}_{w,0}(\Omega)). \tag{9.1.1}$$

As a space of test functions we choose

$$X^{r',q'}_{w',\sigma}(0,T) = \left\{\phi \in X^{r',q'}_{w'}(0,T) \mid \operatorname{div}\phi = 0\right\}.$$

Definition 9.1.1. If f and k are given as in (9.1.1), then a function $u \in L^r(0,T;L^q_w(\Omega))$ is called a very weak solution to the instationary Stokes equations if

$$\begin{aligned} -\langle u, \phi_t\rangle_{\Omega,T} - \langle u, \Delta\phi\rangle_{\Omega,T} &= \langle f, \phi\rangle_{\Omega,T}, &&\text{for every } \phi \in X^{r',q'}_{w',\sigma} \text{ and}\\ -\langle u(t), \nabla\psi\rangle_\Omega &= \langle k(t), \psi\rangle_\Omega, &&\text{for every } \psi \in W^{1,q'}_{w'}(\Omega)\\ &&&\text{and almost every } t \in (0,T). \end{aligned}$$

Note that there does not occur any explicit initial condition $u(0)$. Analogously to the stationary case in Chapter 6 it is hidden implicitly in the definition, since the test functions do not vanish at time $t = 0$. Moreover such explicit boundary conditions would not be reasonable, as shown in the following considerations.

Let $u \in L^r(0,T;L^q_w(\Omega))$. Then

$$\begin{aligned} f &:= [\phi \mapsto \langle u, -\phi_t - \Delta\phi\rangle]\\ &\in \left\{\phi \in W^{1,r'}(0,T;L^{q'}_{w'}(\Omega)) \mid \phi(T) = 0\right\}' + (L^{r'}(0,T;Y^{2,q'}_{w'}(\Omega)))' = (X^{r',q'}_{w'})',\\ k(t) &:= [\psi \mapsto \langle u(t), \nabla\psi\rangle] \in W^{-1,q}_{w,0}(\Omega) \text{ for almost every } t \in (0,T), \end{aligned}$$

and since $\|k(t)\|_{-1,q,w,0} \le \|u(t)\|_{q,w}$ for almost every t, one has $k \in L^r(0,T;W^{-1,q}_{w,0}(\Omega))$. Thus according to Definition 9.1.1 every $u \in L^r(0,T;L^q_w(\Omega))$ is a very weak solution to the instationary Stokes problem with respect to appropriate data.

As in the stationary case we need strong solutions to the instationary Stokes problem to prove existence and uniqueness results for very weak solutions. This has been treated in [26]. More precisely one has:

Theorem 9.1.2. *Let $1 < q < \infty$, $w \in A_q$ and let $\Omega \subset \mathbb{R}^n$ be a bounded $C^{1,1}$-domain. Moreover, let $0 < T \le \infty$. Then for every $f \in L^r(0,T;L^q_w(\Omega))$ there exists a unique solution $u \in L^r(0,T;\mathcal{D}(\mathcal{A}_{q,w})) = L^r(0,T;Y^{2,q}_{w,\sigma}(\Omega))$ with $u_t \in L^r(0,T;L^q_{w,\sigma}(\Omega))$ to the Stokes equations*

$$u_t + \mathcal{A}u = P_{q,w}f \quad \text{a.e. in } (0,T),\ u(0) = 0,$$

where $\mathcal{A}$ is the classical Stokes operator in $L^q_{w,\sigma}(\Omega)$. This solution fulfills the estimate

$$\|u_t\|_{L^r(L^q_{w,\sigma})} + \|\mathcal{A}u\|_{L^r(L^q_{w,\sigma})} \le c\|P_{q,w}f\|_{L^r(L^q_{w,\sigma})},$$

where c is independent of f and T.

To obtain the solvability of the instationary Stokes equations in the very weak sense in Theorem 9.2.1 below, we dualize the strong solutions analogously to the stationary case. For the dualization procedure we need to have them in an appropriate form. In particular, we need the corresponding pressure term because it takes the role of the test function in the divergence equation in the context of very weak solutions. Thus we have to show the existence of a pressure function for strong solutions. This follows from de Rham's Theorem as shown below.

Moreover, we use test functions which assume the "initial condition" $u(T) = 0$ at time T. Since these test functions are related to strong solutions to the Stokes equations, we construct solutions which go backwards in time.

Let $\phi \in L^r(0,T;Y^{2,q}_{w,\sigma}(\Omega)) \cap W^{1,r}(0,T;L^q_w(\Omega))$ be a strong solution to the instationary Stokes problem in the sense of Theorem 9.1.2 with respect to the exterior force $v \in L^r(0,T;L^q_w(\Omega))$. Then one has for every $\eta \in C^\infty_{0,\sigma}(\Omega)$

$$\langle \phi_t(t), \eta\rangle_\Omega - \langle \Delta\phi, \eta\rangle_\Omega - \langle v, \eta\rangle_\Omega = \langle \phi_t + \mathcal{A}\phi - P_{q,w}v, \eta\rangle_\Omega = 0$$

for almost every t. Thus, by de Rham's Theorem [51] there exists a distribution $\psi(t) \in C^\infty_0(\Omega)'$ such that

$$-\Delta\phi(t) + \nabla\psi(t) = v(t) - \phi_t(t)$$

for almost every t. Then from

$$\nabla\psi = \Delta\phi - \phi_t + v \in L^r(0,T;L^q_w(\Omega))$$

and from Lemma 8.1.7 we obtain that $\psi(t) \in W^{1,q}_w(\Omega)$ for alomst every t; if we assume in addition that $\int_\Omega \psi(t) = 0$ for every $t \in (0,T)$ we get $\psi \in L^r(0,T;W^{1,q}_w(\Omega))$ and the estimate

$$\begin{aligned}\|\psi\|_{L^r(W^{1,q}_w)} \le c\|\nabla\psi\|_{L^r(L^q_w)} &\le c\left(\|\mathcal{A}\phi\|_{L^r(L^q_w)} + \|\phi_t\|_{L^r(L^q_w)} + \|v\|_{L^r(L^q_w)}\right)\\ &\le c\|v\|_{L^r(L^q_w)}.\end{aligned}$$

Since we use test functions that vanish at time T instead of 0, we set $\tilde{\phi}(t) := \phi(T-t)$ and $\tilde{\psi}(t) := -\psi(T-t)$. Then we obtain

$$-\tilde{\phi}_t - \Delta\tilde{\phi} - \nabla\tilde{\psi} = \phi_t(T-\cdot) - \Delta\phi(T-\cdot) + \nabla\psi(T-\cdot) = v(T-\cdot)$$

with $\tilde{\phi}(T) = 0$, and $\tilde{\phi}$ and $\tilde{\psi}$ fulfill the estimate

$$\|\tilde{\phi}\|_{X^{r,q}_w} + \|\tilde{\psi}\|_{L^r(W^{1,q}_w)} \le c\|\tilde{\phi}_t\|_{L^r(L^q_w)} + \|\mathcal{A}\tilde{\phi}\|_{L^r(L^q_w)} + \|\nabla\tilde{\psi}\|_{L^r(L^q_w)} \le c\|v\|_{L^r(L^q_w)}, \quad (9.1.2)$$

where we have used

$$\|\tilde{\phi}(t)\|_{W^{2,q}_w} = \|\mathcal{A}^{-1}\mathcal{A}\tilde{\phi}(t)\|_{W^{2,q}_w} \le \|\mathcal{A}\tilde{\phi}(t)\|_{q,w}.$$

9.2 Existence, Uniqueness and Estimates

Theorem 9.2.1. *Let Ω be a bounded $C^{1,1}$-domain and $0 < T \le \infty$. Let f and k be given as in (9.1.1) with $\langle k(t), 1\rangle = 0$ for almost every $t \in (0,T)$.*

Then there exists a unique very weak solution $u \in L^r(0,T;L^q_w(\Omega))$ to the instationary Stokes problem. This function u satisfies the estimate

$$\|u\|_{L^r(L^q_w)} \leq c\left(\|f\|_{(X^{r',q'}_{w'})'} + \|k\|_{L^r(W^{-1,q}_{w,0})}\right) \tag{9.2.1}$$

with a constant $c = c(r,q,w,\Omega) > 0$.

Proof. First assume that $T < \infty$.

As explained in Section 9.1 for every $v \in L^{r'}(0,T;L^{q'}_{w'}(\Omega))$ there exists a unique tuple $(\phi,\psi) \in X^{r',q'}_{w',\sigma} \times L^{r'}(0,T;W^{1,q'}_{w'}(\Omega))$, with

$$-\phi_t - \Delta\phi - \nabla\psi = v.$$

Similarly to the stationary case, we define the solution u by

$$\langle u,v\rangle_{\Omega,T} := \langle f,\phi\rangle_{\Omega,T} + \langle k,\psi\rangle_{\Omega,T}.$$

Then the a priori estimate for the strong solution (9.1.2) implies

$$\begin{aligned} |\langle u,v\rangle_{\Omega,T}| \leq& \|f\|_{(X^{r',q'}_{w'})'}\|\phi\|_{X^{r'q'}_{w'}} + \|k\|_{L^r(W^{-1,q}_{w,0})}\|\psi\|_{L^{r'}(W^{1,q'}_{w'})} \\ \leq& c\left(\|f\|_{(X^{r',q'}_{w'})'} + \|k\|_{L^r(W^{-1,q}_{w,0})}\right)\|v\|_{L^{r'}(L^{q'}_{w'})}. \end{aligned} \tag{9.2.2}$$

Thus we obtain by (2.4.1)

$$\begin{aligned} &u \in \left(L^{r'}(0,T;L^{q'}_{w'}(\Omega))\right)' = L^r(0,T;L^q_w(\Omega)) \text{ with} \\ &\|u\|_{L^r(L^q_w)} \leq c\left(\|f\|_{(X^{r',q'}_{w'})'} + \|k\|_{L^r(W^{-1,q}_{w,0})}\right). \end{aligned}$$

Moreover, for every $(\phi,\psi) \in X^{r',q'}_{w',\sigma} \times L^{r'}(0,T;W^{1,q'}_{w'}(\Omega))$ we have

$$\begin{aligned} -\langle u,\phi_t\rangle_{\Omega,T} - \langle u,\Delta\phi\rangle_{\Omega,T} - \langle u,\nabla\psi\rangle_{\Omega,T} &= -\langle u,\phi_t + \Delta\phi + \nabla\psi\rangle_{\Omega,T} \\ &= \langle f,\phi\rangle_{\Omega,T} + \langle k,\psi\rangle_{\Omega,T}, \end{aligned}$$

where we have used that the mapping

$$v = -\phi_t - \Delta\phi - \nabla\psi \mapsto (\phi,\psi)$$

is well-defined. This shows that u is a very weak solution to the instationary Stokes problem according to Definition 9.1.1 and finishes the proof of existence and a priori estimates.

To show the uniqueness let $U \in L^r(0,T;L^q_w(\Omega))$ be another very weak solution with respect to the data f and k. Moreover, let $v \in L^{r'}(0,T;L^{q'}_{w'}(\Omega))$ and let $\phi \in X^{r',q'}_{w',\sigma}$ and $\psi \in L^{r'}(0,T;W^{1,q'}_{w'}(\Omega))$ solve $v = -\phi_t - \Delta\phi - \nabla\psi$ as above. Then one has

$$\langle U,v\rangle = -\langle U,\phi_t\rangle - \langle U,\Delta\phi\rangle - \langle U,\nabla\psi\rangle = \langle f,\phi\rangle + \langle k,\psi\rangle = \langle u,v\rangle.$$

Since v was arbitrary, this implies $U = u$ and the proof for $T < \infty$ is complete.

For $T = \infty$ we take $v \in L^{r'}(\mathbb{R}_+;L^{q'}_{w'}(\Omega))$, with $\operatorname{supp} v \subset (0,N) \times \overline{\Omega}$ for some $N \in \mathbb{N}$.

Let

$$(\phi,\psi) \in X^{r',q'}_{w',\sigma}(0,N) \times L^{r'}(0,N;W^{1,q'}_{w'}(\Omega))$$

with $\int_\Omega \psi(t) = 0$ for almost every t be the unique solution of

$$-\phi_t - \Delta\phi - \nabla\psi = v, \quad \text{with} \quad \phi(N) = 0. \tag{9.2.3}$$

Extending the functions ϕ and ψ by 0 on $[N,\infty) \times \Omega$ one obtains $\phi \in X^{r',q'}_{w',\sigma}(0,\infty)$ and $\psi \in L^{r'}(0,\infty;W^{1,q'}_{w'}(\Omega))$. Note that $\phi \in W^{1,r'}(0,N;L^{q'}_{w'}(\Omega))$ with $\phi(N) = 0$ yields that the extension of ϕ by 0 to $\mathbb{R}_+$ is contained in $W^{1,r'}(\mathbb{R}_+;L^{q'}_{w'}(\Omega))$.

The functions ϕ and ψ are independent of the choice of N. To see this let $v \in L^{r'}(0,N_1,L^{q'}_{w'}(\Omega))$ and assume it to be extended by 0 to a function in $L^{r'}(0,N_2,L^{q'}_{w'}(\Omega))$, $N_2 > N_1$, again denoted by v. Let

$$(\phi,\psi) \in X^{r',q'}_{w',\sigma}(0,N_1) \times L^{r'}(0,N_1;W^{1,q'}_{w'}(\Omega))$$

be the solutions of (9.2.3). Then their extension by 0 solves the same problem with N_1 replaced by N_2. The uniqueness of the strong solutions to the instationary Stokes equation yields that the solution with respect to N_2 coincides with (ϕ,ψ).

Thus the mapping

$$u := \left[\bigcup_{N=1}^{\infty} L^{r'}(0,N,L^{q'}_{w'}(\Omega)) \ni v \mapsto \langle f,\phi\rangle_{\Omega,\infty} + \langle k,\psi\rangle_{\Omega,\infty}\right]$$

is well-defined, where every $v \in L^{r'}(0,N,L^{q'}_{w'}(\Omega))$ is assumed to be extended by zero to $\mathbb{R}_+$.

We obtain that $u|_{(0,N)} \in L^r(0,N,L^q_w(\Omega))$ for every $N \in \mathbb{N}$. Moreover, since the set of functions with compact support in time is dense in $L^{r'}(0,\infty,L^{q'}_{w'}(\Omega))$ and the estimates in (9.2.2) are independent of T, this yields $u \in L^r(0,\infty;L^q_w(\Omega))$ and the asserted estimate.

The uniqueness in the case $T = \infty$ follows from the uniqueness in the case $T < \infty$: Let U be another very weak solution with respect to the same data f and k. Then $U|_{[0,T]}$ and $u|_{[0,T]}$ are very weak solutions with respect to $f|_{X^{r',q'}_{w'}(0,T)}$ and $k|_{L^{r'}(0,T;W^{1,q'}_{w'}(\Omega))}$ for every $T < \infty$. Thus we obtain $U|_{[0,T]} = u|_{[0,T]}$ for every T proving the uniqueness. □

Corollary 9.2.2. *Assume $f \in L^r(0,T;Y^{-2,q}_w(\Omega))$ and $k \in L^r(0,T;W^{-1,q}_{w,0}(\Omega))$. One has $L^r(0,T;Y^{-2,q}_w(\Omega)) \subset \left(X^{r',q'}_{w'}(0,T)\right)'$ and the associated very weak solution which exists according to Theorem 9.2.1 satisfies the stronger estimate*

$$\left\| u_t|_{Y^{2,q'}_{w',\sigma}(\Omega)} \right\|_{L^r(Y^{-2,q}_{w,\sigma})} + \|u\|_{L^r(L^q_w)} \le c\left(\|f\|_{L^r(Y^{-2,q}_w)} + \|k\|_{L^r(W^{-1,q}_{w,0})}\right) \tag{9.2.4}$$

with $c = c(r,q,w,\Omega) > 0$.

If in addition $k = 0$ then u solves the equation

$$u'|_{Y^{2,q'}_{w',\sigma}(\Omega)} + \mathcal{A}u = f|_{Y^{2,q'}_{w',\sigma}(\Omega)},$$

where $\mathcal{A}$ is the generalized Stokes operator in $Y^{-2,q}_{w,\sigma}(\Omega)$.

Proof. Since $X^{r';q'}_{w'}(0,T) \subset L^{r'}(0,T;Y^{2,q'}_{w'}(\Omega))$, it follows by (2.4.1) that

$$L^r(0,T;Y^{-2,q}_w(\Omega)) = \Big(L^{r'}(0,T;Y^{2,q'}_{w'}(\Omega))\Big)' \subset \Big(X^{r',q'}_{w'}(0,T)\Big)'.$$

This shows the inclusion.

Let $\phi \in C^\infty_0(0,T;Y^{2,q'}_{w',\sigma}(\Omega))$. Then we can estimate using (9.2.1)

$$\begin{aligned} |\langle u_t|_{Y^{2,q'}_{w',\sigma}(\Omega)}, \phi\rangle_{\Omega,T}| &= |\langle u_t, \phi\rangle_{\Omega,T}| = |\langle u, \phi_t\rangle_{\Omega,T}| \\ &\le |\langle u, \Delta\phi\rangle_{\Omega,T}| + |\langle f, \phi\rangle_{\Omega,T}| \\ &\le \|u\|_{L^r(L^q_w)}\|\Delta\phi\|_{L^{r'}(L^{q'}_{w'})} + \|f\|_{L^r(Y^{-2,q}_w)}\|\phi\|_{L^{r'}(Y^{2,q'}_{w'})} \\ &\le c\left(\|f\|_{L^r(Y^{-2,q}_w)} + \|k\|_{L^r(W^{-1,q}_{w,0})}\right)\|\phi\|_{L^{r'}(Y^{2,q'}_{w'})}. \end{aligned}$$

Together with (9.2.1), the a priori estimate in Theorem 9.2.1, this proves the assertion.

The last assertion follows from the characterization of the Stokes operator in Theorem 8.2.3 and the formulation of very weak solutions. □

Remark 9.2.3. As a consequence of Corollary 9.2.2 we obtain that the generalized Stokes operator $\mathcal{A}$ in $Y^{-2,q}_{w,\sigma}(\Omega)$ has maximal regularity defined in Section 2.2.

To see this take $f \in L^r(0,T;Y^{-2,q}_{w,\sigma}(\Omega))$ and $k = 0$. Then the very weak solution $u \in L^r(0,T;L^q_w(\Omega))$ corresponding to any $F \in L^r(0,T;Y^{-2,q}_w(\Omega))$ with $F|_{Y^{2,q'}_{w',\sigma}(\Omega)} = f$ and $k = 0$ exist by Corollary 9.2.2 and solves

$$\partial_t u + \mathcal{A}u = f, \quad u(0) = 0.$$

(The assertion about the initial condition is shown in Lemma 9.2.4 below.) Then by Lemma 11.2.2 below the function u is given by the Variation of Constants formula

$$u(t) = \int_0^t e^{-\mathcal{A}(t-\tau)} f(\tau)d\tau,$$

which means u is a mild solution, cf. Section 2.2. In particular, the mild solution is weakly differentiable in time and takes values in $\mathcal{D}(\mathcal{A}) = L^q_{w,\sigma}(\Omega)$.

This proves that the generalized Stokes operator $\mathcal{A}$ in $Y^{-2,q}_{w,\sigma}(\Omega)$ has maximal regularity.

According to Definition 9.1.1 every $u \in L^r(0,T;L^q_w(\Omega))$ is a very weak solution with respect to appropriate data. This means that such solutions in general do not possess enough time-regularity to ensure that the initial condition $u(0) = u_0$ is well-defined.

However, if the data is chosen as in Corollary 9.2.2, we obtain $u \in L^r(0,T;L^q_w(\Omega))$ and $u_t|_{Y^{2,q'}_{w',\sigma}(\Omega)} \in L^r(0,T;Y^{-2,q}_{w,\sigma}(\Omega))$. As stated in Section 2.4, this regularity suffices to define $u(0)|_{Y^{2,q'}_{w',\sigma}(\Omega)} \in Y^{-2,q}_{w,\sigma}(\Omega)$, and one has

$$\langle u(0), \phi(0)\rangle_\Omega = \langle u, \phi_t\rangle_{\Omega,T} + \langle u_t, \phi\rangle_{\Omega,T} \ \text{ for every } \phi \in C^1_0([0,T], Y^{2,q'}_{w',\sigma}(\Omega)), \quad \phi(T) = 0.$$

Analogously to the case of strong solutions the gradient part of the initial condition cannot be prescribed and is not needed for the uniqueness of the solution.

Lemma 9.2.4. *If* $u \in L^r(0,T;L^q_w(\Omega))$ *is a very weak solution according to Definition 9.1.1 with respect to* $f \in L^r(0,T;Y^{-2,q}_w(\Omega))$ *and* $k \in L^r(0,T;W^{-1,q}_{w,0}(\Omega))$ *then* $u(0)|_{Y^{2,q'}_{w',\sigma}(\Omega)} = 0$.

Proof. For $\phi \in C^1_0((0,T);Y^{2,q'}_{w',\sigma}(\Omega))$ one has

$$\langle u_t, \phi\rangle_{\Omega,T} = -\langle u, \phi_t\rangle_{\Omega,T} = +\langle u, \Delta\phi\rangle_{\Omega,T} + \langle f, \phi\rangle_{\Omega,T}$$

which implies $u_t|_{Y^{2,q'}_{w',\sigma}(\Omega)} \in L^r(0,T;Y^{-2,q}_{w,\sigma}(\Omega))$ and

$$\langle u_t, \phi\rangle_{\Omega,T} = \langle u, \Delta\phi\rangle_{\Omega,T} + \langle f, \phi\rangle_{\Omega,T} = -\langle u, \phi_t\rangle_{\Omega,T} \quad \text{for all } \phi \in X^{r',q'}_{w'},$$

because one can approximate $\phi \in X^{r',q'}_{w'}$ by a sequence in $C^1_0((0,T);Y^{2,q'}_{w'}(\Omega))$ converging in $L^{r'}(0,T;Y^{2,q'}_{w'}(\Omega))$. Thus

$$\langle u(0), \phi(0)\rangle_{\Omega} = \langle u_t, \phi\rangle_{\Omega,T} + \langle u, \phi_t\rangle_{\Omega,T} = 0$$

for every $\phi \in C^1([0,T];Y^{2,q'}_{w',\sigma}(\Omega))$ with $\phi(T) = 0$. In particular, for a fixed $\zeta \in Y^{2,q'}_{w',\sigma}(\Omega)$ and $\eta \in C^\infty_0([0,T))$ with $\eta(0) = 1$ one has $\langle u(0), \zeta\rangle_\Omega = \langle u(0), \zeta\eta(0)\rangle_\Omega = 0$. We have proved $u(0)|_{Y^{2,q'}_{w',\sigma}(\Omega)} = 0$. $\square$

9.3 The Spaces $H^{\beta,r}(X)$

For the treatment of solutions in Bessel potential spaces with inhomogeneous divergence and boundary conditions we need higher time regularity of this part of the data. To measure this time regularity we work in Banach space-valued Bessel potential spaces.

For $\beta \in \mathbb{R}$ we set $\Lambda^\beta_t := \mathcal{F}^{-1}\langle\tau\rangle^\beta\mathcal{F}$, recall that $\langle\tau\rangle^\beta = (1+|\tau|^2)^{\frac{\beta}{2}}$, $\tau \in \mathbb{R}^n$. Using this, for $r > 1$ we define the X-valued Bessel-potential space by

$$H^{\beta,r}(\mathbb{R};X) := \left\{u \in \mathcal{S}'(\mathbb{R};X) \mid \Lambda^\beta_t u \in L^r(\mathbb{R};X)\right\},$$

equipped with the norm

$$\|u\|_{H^{\beta,r}(\mathbb{R};X)} := \|\Lambda^\beta_t u\|_{L^r(\mathbb{R};X)}.$$

Moreover, we define

$$H^{\beta,r}(0,T;X) := \left\{u|_{C^\infty_0(0,T;\mathbb{R})} \mid u \in H^{\beta,r}(\mathbb{R};X)\right\}$$

equipped with the norm

$$\|u\|_{H^{\beta,r}(0,T;X)} := \inf\left\{\|U\|_{H^{\beta,r}(\mathbb{R};X)} \mid U \in H^{\beta,r}(\mathbb{R};X), U|_{C^\infty_0(0,T;\mathbb{R})} = u\right\}.$$

Finally, we set for $\beta \geq 0$

$$H^{\beta,r}_0((0,T];X) := \left\{U|_{C^\infty_0(0,T;\mathbb{R})} \mid U \in H^{\beta,r}(\mathbb{R};X),\ \operatorname{supp} U \subset [0,\infty)\right\}$$

equipped with

$$\|u\|_{H_0^{\beta,r}((0,T];X)} := \inf\big\{\|U\|_{H^{\beta,r}(\mathbb{R};X)} \mid U \in H^{\beta,r}(\mathbb{R};X),$$
$$\operatorname{supp} U \subset [0,\infty),\ U|_{C_0^\infty(0,T;\mathbb{R})} = u\big\}$$

and

$$H_0^{\beta,r}(0,T;X) := \overline{C_0^\infty(0,T;X)}^{H^{\beta,r}(\mathbb{R};X)} \quad \text{with} \quad \|\cdot\|_{H_0^{\beta,r}(0,T;X)} = \|\cdot\|_{H^{\beta,r}(\mathbb{R};X)}.$$

Lemma 9.3.1. *Let X be a reflexive Banach space and $\beta \geq 0$. Then one has*

$$H^{-\beta,r}(\mathbb{R};X) \cong (H^{\beta,r'}(\mathbb{R};X'))' \quad \textit{and} \quad H^{-\beta,r}(0,T;X) \cong (H_0^{\beta,r'}(0,T;X'))'$$

with equivalent norms. Every $u \in H^{-\beta,r}(\mathbb{R};X)$ is identified with the element of the space $(H^{\beta,r'}(\mathbb{R};X'))'$ fulfilling

$$\phi x^* \mapsto \langle u, \phi x^*\rangle_{X,X',\mathbb{R}} := \big\langle \langle u(t), \phi(t)\rangle_{\mathbb{R}}, x^*\big\rangle_{X,X'}, \tag{9.3.1}$$

where $\phi \in \mathcal{S}(\mathbb{R};\mathbb{R})$ and $x^* \in X'$. With this identification one has

$$\langle u, \psi\rangle_{X,X',\mathbb{R}} = \int_{\mathbb{R}} \Big\langle \Lambda_t^{-\beta} u(s), \Lambda_t^{\beta}\psi(s)\Big\rangle_{X,X'} ds$$

for every $u \in H^{-\beta,r}(\mathbb{R};X)$ and $\psi \in H^{\beta,r'}(\mathbb{R};X')$.

Proof. First, we show that for $u \in H^{-\beta,r}(\mathbb{R};X)$ the equation (9.3.1) defines a continuous functional in $(H^{\beta,r'}(\mathbb{R};X'))'$.

The linear hull of

$$\{\phi x^* \mid \phi \in \mathcal{S}(\mathbb{R};\mathbb{R}), \quad x^* \in X'\}$$

is dense in $H^{\beta,r'}(\mathbb{R};X')$. This can be shown for $\beta = 0$, i.e., in the case of $L^{r'}(\mathbb{R};X')$ by the use of the Hahn-Banach theorem using (2.4.1). If $\beta \neq 0$ one uses the isomorphism Λ_t and the fact that Λ_t maps $S(\mathbb{R};\mathbb{R})$ to itself.

Moreover, take $u \in H^{-\beta,r}(\mathbb{R};X)$ and $\phi \in \mathcal{S}(\mathbb{R},\mathbb{R})$, $x^* \in X'$. Then

$$\begin{aligned}\langle u, \phi x^*\rangle_{X,X',\mathbb{R}} &= \big\langle \langle u, \phi\rangle_{\mathbb{R}}, x^*\big\rangle_{X,X'} = \big\langle \langle \Lambda_t^{-\beta} u, \Lambda_t^{\beta}\phi\rangle_{\mathbb{R}}, x^*\big\rangle_{X,X'}\\ &= \int_{\mathbb{R}} \langle \Lambda_t^{-\beta} u(s), \Lambda_t^{\beta}\phi(s)x^*\rangle_{X,X'} ds.\end{aligned}$$

Thus $|\langle u, \phi x^*\rangle_{X,X',\mathbb{R}}| \leq \|u\|_{H^{-\beta,r}(\mathbb{R};X)}\|\phi x^*\|_{H^{\beta,r'}(\mathbb{R};X')}$, and we obtain that $\langle u, \cdot\rangle_{X,X',\mathbb{R}}$ extends in a unique way to a continuous functional on $H^{\beta,r'}(\mathbb{R};X')$ using the density shown above. In particular, this extension fulfills

$$\langle u, \psi\rangle_{X,X',\mathbb{R}} = \int_{\mathbb{R}} \langle \Lambda_t^{-\beta} u(s), \Lambda_t^{\beta}\psi(s)\rangle_{X,X'} ds$$

for every $\psi \in H^{\beta,r'}(\mathbb{R};X')$.

Vice versa let $u \in \big(H^{\beta,r'}(\mathbb{R};X')\big)'$. Then, since X is reflexive, u defines a distribution $u \in \mathcal{S}'(\mathbb{R};X)$ by

$$\mathcal{S}(\mathbb{R};\mathbb{R}) \ni \phi \mapsto [X' \ni x^* \mapsto \langle u, \phi x^*\rangle] \in X'' = X.$$

For $\phi \in \mathcal{S}(\mathbb{R},\mathbb{R})$, $x^* \in X'$ one has

$$\begin{aligned}\left|\left\langle\langle\Lambda_t^{-\beta}u,\phi\rangle_{\mathbb{R}},x^*\right\rangle_{X,X'}\right| = \left|\left\langle\langle u,\Lambda_t^{-\beta}\phi\rangle_{\mathbb{R}},x^*\right\rangle_{X,X'}\right| &\le \|u\|_{(H^{-\beta,r'}(\mathbb{R};X'))'}\|\Lambda_t^{-\beta}\phi x^*\|_{H^{\beta,r'}(\mathbb{R};X')}\\ &= \|u\|_{(H^{-\beta,r'}(\mathbb{R};X'))'}\|\phi x^*\|_{L^{r'}(\mathbb{R};X')}.\end{aligned}$$

Thus by (2.4.1) the functional $\Lambda_t^{-\beta}u$ can be identified with an element of $L^r(\mathbb{R};X)$, or u with an element of $H^{-\beta,r}(\mathbb{R},X)$.

The assertion $H^{-\beta,r}(0,T;X) \cong (H_0^{\beta,r'}(0,T;X'))'$ follows from the assertion on $\mathbb{R}$ as follows.

For $u \in H^{-\beta,r}(0,T;X)$ there exists

$$U \in H^{-\beta,r}(\mathbb{R};X) = \left(H^{\beta,r'}(\mathbb{R};X')\right)' \quad \text{with} \quad U|_{C_0^\infty(0,T)} = u.$$

Thus it follows for $\phi \in C_0^\infty(0,T)$ and $x^* \in X'$

$$\left\langle\langle u,\phi\rangle_T,x^*\right\rangle_{X,X'} = \langle U,\phi x^*\rangle_{X,X',T}.$$

This extends by density and continuity to a functional in $(H_0^{\beta,r'}(0,T;X'))'$.

Vice versa, for $u \in (H_0^{\beta,r'}(0,T;X'))'$ there exists by the Hahn-Banach theorem a functional $U \in (H^{\beta,r'}(\mathbb{R};X'))' \cong H^{-\beta,r}(\mathbb{R};X)$ such that $U|_{H_0^{\beta,r'}(0,T;X')} = u$. Since X is reflexive, one has

$$\left[\mathcal{S}(\mathbb{R};\mathbb{R}) \ni \phi \mapsto [X' \ni x^* \mapsto \langle U,\phi x^*\rangle]\right] \in H^{-\beta,r}(\mathbb{R};X'') = H^{-\beta,r}(\mathbb{R};X)$$

and $U|_{C_0^\infty(0,T)} \in H^{-\beta,r}(0,T;X)$. □

Lemma 9.3.2. *Let X be a UMD-space and $\beta \in \mathbb{R}$. Then*

1. $$\begin{aligned}\partial_t : \quad & H^{\beta,r}(\mathbb{R};X) & \to H^{\beta-1,r}(\mathbb{R};X),\\ \partial_t : \quad & H^{\beta,r}(0,T;X) & \to H^{\beta-1,r}(0,T;X)\\ \partial_t : \quad & H_0^{\beta,r}((0,T];X) & \to H_0^{\beta-1,r}((0,T];X)\end{aligned}$$

   is continuous.

2. *For $k \in \mathbb{Z}$ one has $H^{k,r}(\mathbb{R};X) \cong W^{k,r}(\mathbb{R};X)$ and $H^{k,r}(0,T;X) \cong W^{k,r}(0,T;X)$ with equivalent norms.*

   The isomorphism is given by the identification in (9.3.1).

3. *Let $\beta \in [0,1]$ and let X_1, X_2 be UMD-spaces with $X_1 \hookrightarrow X_2$. Then there exists a continuous linear extension operator*

   $$E : H_0^{\beta,r}((0,T];X_2) \cap L^r(0,T;X_1) \to H^{\beta,r}(\mathbb{R};X_2) \cap L^r(\mathbb{R};X_1)$$

   with $Eu(t) = 0$ for every $t < 0$.

Proof. Since X is a UMD-space, it follows from Theorem 2.4.2 that the operator

$$u \mapsto \mathcal{F}^{-1}\frac{t}{\langle t\rangle}\hat{u} : \quad L^r(\mathbb{R},X) \to L^r(\mathbb{R},X)$$

is continuous.
1. For $u \in H^{\beta,r}(\mathbb{R};X)$ it holds

$$\begin{aligned}\|\partial_t u\|_{H^{\beta-1,r}(\mathbb{R},X)} &= \|\mathcal{F}^{-1}t\langle t\rangle^{\beta-1}\hat{u}\|_{L^r(\mathbb{R},X)} \\ &= \|\mathcal{F}^{-1}\frac{t}{\langle t\rangle}\mathcal{F}\mathcal{F}^{-1}\langle t\rangle^{\beta}\hat{u}\|_{L^r(\mathbb{R},X)} \le c\|u\|_{H^{\beta,r}(0,T;X)}.\end{aligned}$$

This proves the assertion for $H^{\beta,r}(\mathbb{R},X)$.

Now let $u \in H^{\beta,r}(0,T;X)$ and let $U \in H^{\beta,r}(\mathbb{R};X)$ be an extension of u with $\|U\|_{H^{\beta,r}(\mathbb{R};X)} \le 2\|u\|_{H^{\beta,r}(0,T;X)}$. Then $\partial_t U$ is an extension of $\partial_t u$. Thus

$$\|\partial_t u\|_{H^{\beta-1,r}(0,T;X)} \le \|\partial_t U\|_{H^{\beta-1,r}(\mathbb{R};X)} \le c\|U\|_{H^{\beta,r}(\mathbb{R};X)} \le 2c\|u\|_{H^{\beta,r}(0,T;X)}.$$

The assertion for $H_0^{\beta,r}((0,T];X)$ is proved analogously.
2. We start with the case $k \ge 0$. As above one uses Theorem 2.4.2 to show that $\frac{t^j}{\langle t\rangle^k}$, $j = 0,...,k$, is a multiplier and hence

$$\|\partial_t^j u\|_{L^r(\mathbb{R};X)} = \left\|\mathcal{F}^{-1}\frac{t^j}{\langle t\rangle^k}\mathcal{F}^{-1}\mathcal{F}\langle t\rangle^k\hat{u}\right\|_{L^r(\mathbb{R};X)} \le c\|u\|_{H^{k,r}(\mathbb{R};X)}.$$

For the opposite direction one uses the decomposition

$$\langle t\rangle = \frac{1+t^2}{\langle t\rangle} \quad \text{and} \quad \langle t\rangle^k = \sum_{j=0}^{k}\binom{k}{j}\frac{t^j}{\langle t\rangle^k}t^j$$

as well as the fact that $\frac{t^j}{\langle t\rangle^k}$ is a multiplier for $j = 0,...,k$.

This proves $H^{k,r}(\mathbb{R};X) \cong W^{k,r}(\mathbb{R};X)$, $k \ge 0$. By the duality shown in Theorem 9.3.1 we obtain the same result for $k < 0$.

For $u \in W^{k,r}(0,T;X)$, $k > 0$, we construct an extension

$$Eu(x) = \begin{cases} \phi(-x)\sum_{j=1}^{k+1}\lambda_j u(-jx) & \text{if } -\frac{T}{k+1} < x < 0, \\ u(x) & \text{if } x \in [0,T], \\ \phi(x-T)\sum_{j=1}^{k+1}\lambda_j u(T - j\cdot(x-T)) & \text{if } T < x < T + \frac{T}{k+1}, \\ 0 & \text{else,} \end{cases} \tag{9.3.2}$$

where ϕ is a smooth cut-off function with $\phi = 0$ in a neighborhood of $\frac{T}{k+1}$, $\phi = 1$ in a neighborhood of 0 and with $\sum_j \lambda_j(-j)^l = 1$ for $l = 0,...,k$.

Thus for $u \in W^{k,r}(0,T;X)$ one has $Eu \in W^{k,r}(\mathbb{R};X) = H^{k,r}(\mathbb{R};X)$ which shows that $u \in H^{k,r}(0,T;X)$ with

$$\|u\|_{H^{k,r}(0,T;X)} \le \|Eu\|_{H^{k,r}(\mathbb{R};X)} \le c\|Eu\|_{W^{k,r}(\mathbb{R};X)} \le c\|u\|_{W^{k,r}(0,T;X)}.$$

Vice versa for $u \in H^{k,r}(0,T;X)$ an appropriate extension exists by definition. Hence an analogous argument completes the proof for $k \geq 0$.

For $k < 0$ the assertion follows by duality stated in Lemma 9.3.1 since

$$\begin{aligned} H^{k,r}(0,T;X) &\cong \left[\overline{(C_0^\infty(\mathbb{R};X))}^{H^{-k,r'}(\mathbb{R};X')}\right]' \\ &= \left[\overline{(C_0^\infty(\mathbb{R};X))}^{W^{-k,r'}(\mathbb{R};X')}\right]' = W^{k,r}(0,T;X). \end{aligned}$$

3. We begin to define the extension by 0 to the negative half axis by

$$\begin{aligned} E_0 : H_0^{\beta,r}((0,T];X_2) \cap L^r(0,T;X_1) \to & H^{\beta,r}(-\infty,T;X_2) \cap L^r(-\infty,T;X_1), \\ & u \mapsto \begin{cases} u & \text{on } (0,T) \\ 0 & \text{on } (-\infty,0) \end{cases} \end{aligned}$$

which is continuous by the definition of $H_0^{\beta,r}((0,T];X_2)$. Moreover, by E we denote the extension to $t > T$ defined in the same way as in (9.3.2) with $k = 1$. By construction

$$\begin{aligned} E: \ & L^r(-\infty,T;X_i) && \to L^r(\mathbb{R};X_i), \quad i = 1,2 \ \text{ and} \\ E: \ & H^{1,r}(-\infty,T;X_2) && \to H^{1,r}(\mathbb{R};X_2) \end{aligned}$$

is continuous. Since X_2, is a UMD-space one has

$$\left[L^r(-\infty;X_2), H^{1,r}(-\infty,T;X_2)\right]_\beta = H^{\beta,r}(-\infty,T;X_2).$$

This is proved in the same way as in the scalar-valued case, cf. [50] 13, Prop. 6.2, replacing the scalar-valued multiplier theorem by the Banach space-valued version in Theorem 2.4.2.

Thus the assertion follows by interpolation. □

9.4 Inhomogeneous Tangential Boundary Conditions

Our next aim is to develop a solution theory of the instationary Stokes equations in weighted Bessel potential spaces. In the context of lowest regularity, in which the class of solutions is contained in $L^r(0,T;L^q_w(\Omega))$ the data could be chosen fully inhomogeneous. Now, turning to higher regularity, we do not want to loose this possibility. However, this requires a more complex theory and stronger assumptions to the time regularity than before.

We start with purely tangential boundary conditions. If $g(t) \in T_w^{\beta,q}(\Omega)$ for almost every t, this means

$$\begin{aligned} g(t,x)\cdot N &= 0 \quad \text{for almost every } \ x \in \partial\Omega \ \text{ if } \ \beta \in [1,2] \ \text{ and} \\ \langle g(t), Nh\rangle_{\partial\Omega} &= 0 \quad \text{for every scalar-valued } \ h \in C^\infty(\overline{\Omega})|_{\partial\Omega} \ \text{ if } \ \beta \in [0,1]. \end{aligned}$$

The reason why we deal with tangential boundary data is that such data can be represented by

$$f := \left[Y^{2,q'}_{w',\sigma}(\Omega) \ni \phi \mapsto \langle g(t), N\cdot\nabla\phi\rangle_{\partial\Omega}\right] \in Y^{-2,q}_{w,\sigma}(\Omega). \tag{9.4.1}$$

In the latter space we have defined the generalized Stokes operator $\mathcal{A}$, see Section 8.2. By Proposition 6.3.7 we obtain that for every f defined as in (9.4.1) one has $\mathcal{A}^{-1}f|_{\partial\Omega} = g$ in the sense of (6.3.6), since $\mathcal{A}^{-1}f$ is a very weak solution to the Stokes equation with exterior force f and $k = 0$.

In this Section the basic tool is the operator-valued Multiplier Theorem 2.4.5. Thus the following Lemma is necessary.

Lemma 9.4.1. *$H_w^{\beta,q}(\Omega)$ is a UMD-space for every $\beta \geq 0$ and $T_w^{\beta,q}(\partial\Omega)$ is a UMD-space for $\beta \in [0, 2]$.*

Proof. By [2, Theorem 4.5.2] the space $L_w^q(\Omega)$ is a UMD-space.

If $k \in \mathbb{N}$ then it is easy to see that $W_w^{k,q}(\Omega)$ is isomorphic to a closed subspace of $L_w^q(\Omega)^N$, $N = |\{\alpha \in N_0^n \mid |\alpha| \leq k\}|$ via the embedding

$$W_w^{k,q}(\Omega) \ni u \mapsto (\partial^\alpha u)_{\alpha\in\mathbb{N}_0^n, |\alpha|\leq k} \in L_w^q(\Omega)^N.$$

For $\beta \in (0, k)$ we use that by [2] the interpolation space $H_w^{\beta,q}(\Omega) = [L_w^q(\Omega), W_w^{k,q}(\Omega)]_{\frac{\beta}{k}}$ is a UMD-space.

For $\beta \in [1, 2]$ the space $T_w^{\beta,q}(\partial\Omega)$ is a UMD-space since it is the factor space of two UMD-spaces [2], more precisely,

$$T_w^{\beta,q}(\partial\Omega) = H_w^{\beta,q}(\Omega)/\{u \in H_w^{\beta,q}(\Omega) \mid u|_{\partial\Omega} = 0\}.$$

For $\beta \in [0, 1)$ one has by definition, see Chapter 8 (8.1.4)

$$T_w^{\beta,q}(\partial\Omega) = \left[T_w^{0,q}(\partial\Omega), T_w^{1,q}(\partial\Omega)\right]_\beta.$$

Thus all that remains to show is that $T_w^{0,q}(\partial\Omega)$ is a UMD-space. This is an easy consequence of Corollary 4.1.6, where it has been shown that

$$T_w^{0,q}(\partial\Omega) \cong \{u \in L_w^q(\Omega) \mid \Delta u = 0\},$$

which is a closed subspace of $L_w^q(\Omega)$. $\square$

The spaces $H_w^{\beta,q}(\Omega)$ and $Y_w^{\beta,q}(\Omega)$ are independent of the values of the weight function outside Ω and the corresponding norms are equivalent. However, we do not know whether the equivalence constants depend A_q-consistently on the weight function. To fix notation and to ensure that interpolation preserves the A_q-consistence of the constants we assume for the rest of this section that the norm on $H_w^{\beta,q}(\Omega)$ is given by the norm in the interpolation space, i.e.

$$\|\cdot\|_{H_w^{\beta,q}(\Omega)} = \|\cdot\|_{[W_w^{k,q}(\Omega), W_w^{k+1,q}(\Omega)]_\theta}, \text{ where } \beta \in [k, k+1] \text{ and } \theta = \beta - k.$$

Accordingly we assume $\|\cdot\|_{Y_w^{\beta,q}(\Omega)} = \|\cdot\|_{H_w^{\beta,q}(\Omega)}$ for $\beta \in [1, 2]$ and

$$\|\cdot\|_{Y_w^{\beta,q}(\Omega)} = \|\cdot\|_{[L_w^q(\Omega), W_{w,0}^{1,q}(\Omega)]_\beta} \quad \text{for} \quad \beta \in [0, 1).$$

In particular $H_w^{k,q}(\Omega)$ is equipped with the norm in $W_w^{k,q}(\Omega)$ for every $k \in \mathbb{N}_0$.

Theorem 9.4.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain of class $C^{1,1}$ and let I be an interval.*

1. *For $2 \geq \beta \geq 0$ let $B(t) : L^q_w(\Omega) \to H^{\beta,q}_w(\Omega)$, $t \in I$, be uniformly bounded for every $w \in A_q$ with an A_q-consistent bound of the continuity constant.*

   Then $B(t)$, $t \in I$, is R-bounded.

2. *The assertion of 1. holds true if one replaces $H^{\beta,q}_w(\Omega)$ by $Y^{\beta,q}_w(\Omega)$.*

Proof. 1. We begin with the case $0 \leq \beta < 1$. Let $(\psi_j)_{j=1}^N$, $\psi_j : \mathbb{R}^n_+ \supset U_j \to V_j \subset \overline{\Omega}$ be a collection of $C^{1,1}$-charts and assume that each ψ_j is extended to a $C^{1,1}$-diffeomorphism from $\mathbb{R}^n$ to $\mathbb{R}^n$. Let $(\phi_j)_j$ be a decomposition of unity subordinate to the covering $(V_j)_j$ of $\overline{\Omega}$.

For $v \in A_q$ we set $w_j := v \circ \psi_j^{-1}$ and by $E_e : H^{\beta,q}_v(\mathbb{R}^n_+) \to H^{\beta,q}_{v^*}(\mathbb{R}^n)$ we denote the even extension

$$E_e u(x) = \begin{cases} u(x) & \text{for } x_n \geq 0 \\ u(x', -x_n) & \text{for } x_n \leq 0 \end{cases} \quad \text{for } u \in H^{\beta,q}_v(\mathbb{R}^n_+)$$

and by $E_0 : L^q_w(\Omega) \to L^q_w(\mathbb{R}^n)$ we denote the extension by 0. We consider the mapping $M_j(t) : L^q_v(\mathbb{R}^n) \to L^q_v(\mathbb{R}^n)$, which is defined by the composition

$$\begin{aligned} M_j(t) : L^q_v(\mathbb{R}^n) &\xrightarrow{C_{\psi_j^{-1}} : h \mapsto h \circ \psi_j^{-1}} L^q_{w_j}(\mathbb{R}^n) \xrightarrow{R_\Omega} L^q_{w_j}(\Omega) \xrightarrow{B(t)} H^{\beta,q}_{w_j}(\Omega) \\ &\xrightarrow{M_{\phi_j} : h \mapsto \phi_j h} H^{\beta,q}_{w_j}(H_{\psi_j}) \xrightarrow{C_{\psi_j} : h \mapsto h \circ \psi_j} H^{\beta,q}_v(\mathbb{R}^n_+) \xrightarrow{E_e} H^{\beta,q}_{v^*}(\mathbb{R}^n) \\ &\xrightarrow{\Lambda^\beta} L^q_{v^*}(\mathbb{R}^n) \xrightarrow{R_{\mathbb{R}^n_+}} L^q_v(\mathbb{R}^n_+) \xrightarrow{E_0} L^q_v(\mathbb{R}^n), \end{aligned}$$

where H_{ψ_j} is the bent half space with boundary $\psi_j(\mathbb{R}^{n-1} \times \{0\})$. This operator $M_j(t)$ is the coposition of $B(t)$ with operators constant in t and by Corollary 7.1.3 and the Lemmas 7.2.1, 7.2.2 and 7.2.3 with norms depending A_q-consistently on the weight functions v and w. By the assumptions on $B(t)$ we obtain that $M_j(t)$ is uniformly bounded in t with an A_q-consistent bound. Thus by [26, Theorem 4.3] we obtain that $M_j(t)$ is R-bounded in t.

Next we show that

$$B(t) = \sum_{j=1}^n M_{\tilde{\phi}_j} \circ C_{\psi_j^{-1}} \circ R_{\mathbb{R}^n_+} \circ \Lambda^{-\beta} \circ E_e \circ R_{\mathbb{R}^n_+} \circ M_j(t) \circ C_{\psi_j} \circ E_0, \tag{9.4.2}$$

where

$$M_{\tilde{\phi}_j} : H^{\beta,q}_{(w \circ \psi_j)^* \circ \psi_j^{-1}}(H_{\psi_j}) \to H^{\beta,q}_w(\Omega)$$

is the multiplication with some cut-off function $\tilde{\phi}_j \in C_0^\infty(V_j)$ with $\tilde{\phi}_j \equiv 1$ on $\operatorname{supp} \phi_j$. Clearly

$$R_\Omega \circ C_{\psi_j^{-1}} \circ C_{\psi_j} \circ E_0 = \mathrm{id}_{L^q_w(\Omega)}.$$

Moreover,

$$\begin{aligned} &\sum_{j=1}^N M_{\tilde{\phi}_j} \circ C_{\psi_j^{-1}} \circ \underbrace{R_{\mathbb{R}^n_+} \circ \Lambda^{-\beta} \circ E_e \circ R_{\mathbb{R}^n_+} \circ E_0 \circ R_{\mathbb{R}^n_+} \circ \Lambda^\beta \circ E_e}_{= \mathrm{id}_{H^{\beta,q}_w(\mathbb{R}^n_+)}, \text{ since } \Lambda^\beta \circ E_e \text{ is even}} \circ C_{\psi_j} \circ M_{\phi_j} \\ &= \sum_{j=1}^N M_{\phi_j} = \mathrm{id}_{H^{\beta,q}_w(\Omega)}. \end{aligned}$$

We have used that the Fourier transform and the inverse Fourier transform as well as the multiplication with the even function $\langle\xi\rangle^\beta$ maps even functions to even functions. This shows that the image space of $\Lambda^\beta \circ E_e$ is even. Thus (9.4.2) holds.

Using Lemma 2.4.4 we find that $B(t)$ is R-bounded as a sum and comosition of the R-bounded operators $M_j(t)$ with bounded operators which are constant in t.

We turn to the case $1 < \beta \leq 2$. If $B(t) : L^q_w(\Omega) \to H^{\beta,q}_w(\Omega)$, $t \in I$, fulfills the assumptions of this theorem, then $\partial_i B(t) : L^q_w(\Omega) \to H^{\beta-1,q}_w(\Omega)$ is uniformly bounded for $i = 1, ..., n$ as well, by a constant depending A_q-conistently on w. Moreover, by the embedding $H^{\beta,q}_w(\Omega) \hookrightarrow H^{\beta-1,q}_w(\Omega)$ the same is true for $B(t) : L^q_w(\Omega) \to H^{\beta-1,q}_w(\Omega)$.

Since $0 < \beta - 1 \leq 1$, we are in the case just treated and we find that

$$\partial_i B(t) : L^q_w(\Omega) \to H^{\beta-1,q}_w(\Omega), \quad i = 1, ..., n, \quad \text{and} \quad B(t) : L^q_w(\Omega) \to H^{\beta-1,q}_w(\Omega)$$

are R-bounded. Thus using the notation of Definition 2.4.3 we find by Lemma 7.2.5

$$\begin{aligned}
&\int_0^1 \left\| \sum_{k=1}^m r_k(u) B(t_k) h_k \right\|_{H^{\beta,q}_w(\Omega)} du \\
&\leq c \left(\sum_{j=1}^n \int_0^1 \left\| \partial_j \sum_{k=1}^m r_k(u) B(t_k) h_k \right\|_{H^{\beta-1,q}_w(\Omega)} du + \int_0^1 \left\| \sum_{k=1}^m r_k(u) B(t_k) h_k \right\|_{H^{\beta-1,q}_w(\Omega)} du \right) \\
&\leq c \int_0^1 \left\| \sum_{k=1}^m r_k(u) h_k \right\|_{L^q_w(\Omega)} du.
\end{aligned}$$

Hence $B(t)$ is R-bounded.

2. For $1 \leq \beta \leq 2$ one has $\|\cdot\|_{Y^{\beta,q}_w} = \|\cdot\|_{\beta,q,w}$. Thus, if $B(t) : L^q_w(\Omega) \to Y^{\beta,q}_w(\Omega) \subset H^{\beta,q}_w(\Omega)$ fulfills the assumptions of the theorem, then $B(t) : L^q_w(\Omega) \to H^{\beta,q}_w(\Omega)$ is R-bounded. Since $B(t)$ takes values in $Y^{\beta,q}_w(\Omega)$, we obtain the asserted R-boundedness of $B(t) : L^q_w(\Omega) \to Y^{\beta,q}_w(\Omega)$.

Now we assume $0 \leq \beta < 1$. We choose some ball B_r such that $\overline{\Omega} \subset B_r$. Then the operator

$$E_{0,B_r} : Y^{\beta,q}_w(\Omega) \to H^{\beta,q}_w(B_r), \quad E_{0,B_r}(u)(x) = \begin{cases} u(x) & \text{if } x \in \Omega \\ 0 & \text{if } x \in B_r \setminus \Omega \end{cases} \tag{9.4.3}$$

is continuous with continuity constant 1. This is clear for $\beta = 0$ and $\beta = 1$, for $\beta \in (0,1)$ it follows by interpolation.

We set

$$D(t) : L^q_w(B_r) \to H^{\beta,q}_w(B_r), \quad D(t)u = E_{0,B_r} \circ B(t) \circ R_\Omega,$$

where R_Ω is the restriction to Ω. Then $D(t)$ is uniformly bounded by a constant depending A_q-consistently on w. Hence it is R-bounded by 1.

Now we would like to conclude the R-boundedness of $B(t)$ as before from the R-boundedness of $D(t)$ using $B(t) = R_\Omega \circ D(t) \circ E_{0,B_r}$.

However, to repeat the above procedure we do need for this step is the continuity of E_{0,B_r} and R_Ω. The one of E_{0,B_r} is has been shown in (9.4.3). But the restriction

$$R_\Omega : H^{\beta,q}_w(B_r) \to Y^{\beta,q}_w(\Omega)$$

is in general not well-defined since functions in $H^{\beta,q}_w(B_r)$ in general do not vanish on the boundary of Ω. Thus we use the additional information that functions in the image spaces of E_{0,B_r} and of $D(t)$ already vanish outside Ω. This gives us the estimates we need as follows.

Let $u \in H^{\beta,q}_w(B_r)$ with $u|_{B_r\setminus\Omega} = 0$. Then by the Theorems 7.4.2 and 7.1.2 the norm in the interpolation space is equivalent to the one defined by restrictions. The constants are maybe no longer A_q-consistent, but in this step of the proof this is no longer needed.

Thus we may estimate, denoting by $E_{0,\mathbb{R}^n}$ the extension by 0 to the whole space $\mathbb{R}^n$,

$$\begin{aligned}\|R_\Omega u\|_{Y^{\beta,q}_w(\Omega)} \leq& c\|E_{0,\mathbb{R}^n} R_\Omega u\|_{H^{\beta,q}_w(\mathbb{R}^n)}\\ =& c\|\psi U\|_{H^{\beta,q}_w(\mathbb{R}^n)} \leq c\|U\|_{H^{\beta,q}_w(\mathbb{R}^n)} \leq c\|u\|_{H^{\beta,q}_w(B_r)},\end{aligned}$$

where ψ is some cut-off function with $\operatorname{supp}\psi \subset B_r$ and $\psi = 1$ in Ω and $U \in H^{\beta,q}_w(\mathbb{R}^n)$ is some extension of $(E_{0,\mathbb{R}^n} R_\Omega u)|_{B_r} = u$ with $\|U\|_{H^{\beta,q}_w(\mathbb{R}^n)} \leq c\|u\|_{H^{\beta,q}_w(B_r)}$.

Now the R-boundedness of $B(t)$ follows from the R-boundedness of $D(t)$ as follows.

$$\begin{aligned}\int_0^1 \left\|\sum_{k=1}^m r_k(u)B(t_k)h_k\right\|_{Y^{\beta,q}_w(\Omega)} du =& \int_0^1 \left\|\sum_{k=1}^m r_k(u)R_\Omega \circ D(t_k)\circ E_{0,B_r}h_k\right\|_{Y^{\beta,q}_w(\Omega)} du\\ \leq& c\int_0^1 \left\|\sum_{k=1}^m r_k(u)D(t_k)\circ E_{0,B_r}h_k\right\|_{H^{\beta,q}_w(B_r)} du\\ \leq& c\int_0^1 \left\|\sum_{k=1}^m r_k(u)E_{0,B_r}h_k\right\|_{L^q_w(B_r)} du\\ \leq& c\int_0^1 \left\|\sum_{k=1}^m r_k(u)h_k\right\|_{L^q_w(\Omega)} du.\end{aligned}$$

This completes the proof. □

Lemma 9.4.3. *Let $0 < \delta < \varepsilon$, $\varepsilon \in (0, \frac{\pi}{2})$ and $w \in A_q$. Then the operator*

$$\langle\lambda\rangle^{1-\frac{\beta}{2}}(\lambda + \mathcal{A})^{-1} : L^q_{w,\sigma}(\Omega) \to Y^{\beta,q}_w(\Omega)$$

is bounded uniformly with respect to $\lambda \in \Sigma_\delta \cup \{0\}$. This uniform bound depends A_q-consistently om w.

Proof. For the cases $\beta = 0$ and $\beta = 2$ we observe that by Theorem 6.2.2 the strong solution u of

$$(\lambda + \mathcal{A})u = f$$

fulfills the estimate

$$|\lambda|\|u\|_{q,w} + \|u\|_{2,q,w} \leq c\|f\|_{q,w}$$

with c depending A_q-consistently on w. This yields

$$\|u\|_{2,q,w} \leq c\|f\|_{q,w},$$

which is the assertion for $\beta = 2$ and

$$\langle\lambda\rangle\|u\|_{q,w} \le c(|\lambda|+1)\|u\|_{q,w} \le c\|f\|_{q,w},$$

which is the assertion for $\beta = 0$. Thus we have shown

$$\|(\lambda+\mathcal{A})^{-1}\|_{\mathcal{L}(L^q_{w,\sigma},H^{\beta,q}_w)} \le c\langle\lambda\rangle^{\frac{\beta}{2}-1} \quad \text{for} \quad \beta = 0, 2.$$

Next we consider the case $\beta = 1$. By interpolation we obtain

$$\|(\lambda+\mathcal{A})^{-1}\|_{\mathcal{L}(L^q_{w,\sigma},[L^q_w,H^{2,q}_w]_{\frac{1}{2}})} \le c^{1-\frac{1}{2}}\langle\lambda\rangle^{-(1-\frac{1}{2})}c^{\frac{1}{2}} = c\langle\lambda\rangle^{-\frac{1}{2}},$$

where c is independent of λ and depends A_q-consistently on w. Now Lemma 7.2.4 yields

$$\|(\lambda+\mathcal{A})^{-1}f\|_{Y^{1,q}_w} = \|(\lambda+\mathcal{A})^{-1}f\|_{W^{1,q}_w} \le M\|(\lambda+\mathcal{A})^{-1}f\|_{[L^q_w,H^{2,q}_w]_{\frac{1}{2}}} \le cM\langle\lambda\rangle^{-\frac{1}{2}}\|f\|_{q,w}.$$

This is the assertion for $\beta = 1$. For arbitrary $\beta \in [0,2]$ we use interpolation:
For $0 < \beta < 1$ one has

$$\|(\lambda+\mathcal{A})^{-1}\|_{\mathcal{L}(L^q_{w,\sigma},Y^{\beta,q}_w)} \le c^{1-\beta}\langle\lambda\rangle^{-1+\beta}c^{\beta}\langle\lambda\rangle^{-\beta\cdot\frac{1}{2}} = c\langle\lambda\rangle^{\frac{\beta}{2}-1}$$

and analogously for $1 < \beta < 2$ one has

$$\begin{aligned}\|(\lambda+\mathcal{A})^{-1}\|_{\mathcal{L}(L^q_{w,\sigma},Y^{\beta,q}_w)} &= \|(\lambda+\mathcal{A})^{-1}\|_{\mathcal{L}(L^q_{w,\sigma},H^{\beta,q}_w)} \\ &\le c^{1-(\beta-1)}\langle\lambda\rangle^{-\frac{1}{2}(1-(\beta-1))}c^{\beta-1} = c\langle\lambda\rangle^{\frac{\beta}{2}-1},\end{aligned}$$

where all constants are independent of λ and depend A_q-consistently on w. □

We obtain the following regularity result in the case of purely tangential boundary conditions.

Lemma 9.4.4. *Let* $0 \le \beta \le 2$ *and*

$$g \in L^r(0,T;T^{\beta,q}_w(\partial\Omega)) \cap H_0^{\frac{\beta}{2},r}((0,T];T^{0,q}_w(\partial\Omega))$$

be purely tangential. Let $u \in L^r(0,T;L^q_w(\Omega))$ *be the unique very weak solution to the instationary Stokes problem with zero initial values, force and divergence and boundary condition* g*, i.e.,*

$$\begin{aligned} -\langle u,\partial_t\phi\rangle_{\Omega,T} - \langle u,\Delta\phi\rangle_{\Omega,T} &= -\langle g, N\cdot\nabla\phi\rangle_{\partial\Omega,T} \qquad \textit{for all} \ \ \phi \in X^{r',q'}_{w',\sigma} \\ \langle u(t),\psi\rangle_\Omega &= 0 \qquad\qquad\qquad\qquad\quad \textit{for all} \ \ \psi \in W^{1,q'}_{w'}(\Omega) \end{aligned} \tag{9.4.4}$$

and almost every t.

Then $u \in L^r(0,T;H^{\beta,q}_w(\Omega))$ *and it fulfills the estimate*

$$\|u_t|_{Y^{2,q'}_{w',\sigma}(\Omega)}\|_{L^r(Y^{\beta-2,q}_{w,\sigma})} + \|u\|_{L^r(H^{\beta,q}_w)} \le c\left(\|g\|_{L^r(T^{\beta,q}_w)} + \|g\|_{H_0^{\frac{\beta}{2},r}((0,T];T^{0,q}_w)}\right),$$

with $c = c(r,\Omega,q,A_q(w)) > 0$.

Proof. By the Lemmas 9.3.2 and 9.4.1 we may assume that g is extended to an element of $L^r(\mathbb{R}; T_w^{\beta,q}(\partial\Omega)) \cap H^{\frac{\beta}{2},r}(\mathbb{R}; T_w^{0,q}(\partial\Omega))$ with $g(t) = 0$ for $t < 0$. This is possible without increasing the magnitude of the norm of g. The extension is again denoted by g. Let

$$\begin{aligned} B : \{g \in T_w^{\beta,q}(\partial\Omega) \mid g \text{ purely tangential}\} \to & Y_w^{-2,q}(\Omega), \\ g \mapsto & [\phi \mapsto -\langle g, N \cdot \nabla\phi\rangle]. \end{aligned}$$

Let $u \in L^r(\mathbb{R}; L^q_w(\Omega))$ with $u(t) = 0$ for $t < 0$ and such that, for $t \geq 0$, it is the very weak solution to the instationary Stokes problem with exterior force Bg, for the extended function g. This solution exists by Theorem 9.2.1, is uniquely defined by g and solves the Stokes equations in the sense of (9.4.4) with T replaced by ∞. Moreover, by the uniqueness of very weak solutions, this function u coincides on $[0, T]$ with the very weak solution with respect to the original g, given in the assumption of this theorem.

We have to show that it satisfies $u \in L^r(\mathbb{R}; H_w^{\beta,q}(\Omega))$ and fulfills the estimate.

Set $u_1(t) := \mathcal{A}^{-1}Bg(t)$, where $\mathcal{A}$ is the generalized Stokes operator on $Y^{-2,q}_{w,\sigma}(\Omega)$. Then $u_1(t)|_{\partial\Omega} = g(t)$ in the sense of (6.3.6) for almost every t since g is purely tangential.

Since $\mathcal{A}^{-1}B : T_w^{0,q}(\partial\Omega) \to L^q_w(\Omega)$ is continuous, one obtains

$$\begin{aligned} \|u_1\|_{H^{\frac{\beta}{2},r}(\mathbb{R};L^q_w)} &= \|\Lambda_t^{\frac{\beta}{2}} \mathcal{A}^{-1}Bg\|_{L^r(\mathbb{R};L^q_w)} \\ &= \|\mathcal{A}^{-1}B\Lambda_t^{\frac{\beta}{2}} g\|_{L^r(\mathbb{R};L^q_w)} \leq c\|\Lambda_t^{\frac{\beta}{2}} g\|_{L^r(\mathbb{R};T_w^{0,q})} \\ &= c\|g\|_{H^{\frac{\beta}{2},r}(\mathbb{R};T_w^{0,q})}. \end{aligned}$$

Moreover, from the pointwise estimate in Theorems 8.1.3 and 8.1.4 we obtain $u_1 \in L^r(\mathbb{R}; H_w^{\beta,q}(\Omega))$ and the estimate $\|u_1\|_{L^r(\mathbb{R};H_w^{\beta,q})} \leq c\|g\|_{L^r(\mathbb{R};T_w^{\beta,q})}$. Now $u_2 := u - u_1$ solves

$$\partial_t u_2 + \mathcal{A}u_2 = -\partial_t u_1 \quad \text{in} \quad \mathcal{D}'(\mathbb{R}, Y^{-2,q}_{w,\sigma}(\Omega)).$$

An application of the Fourier transformation with respect to the time variable t yields

$$(it + \mathcal{A})\hat{u}_2 = -it\hat{u}_1 \quad \text{or} \quad \hat{u}_2 = -it(it + \mathcal{A})^{-1}\hat{u}_1.$$

As a next step we show that

$$M(t) := \langle t\rangle^{-\frac{\beta}{2}} t(it + \mathcal{A})^{-1}P_{q,w} \in \mathcal{L}(L^q_w(\Omega), Y_w^{\beta,q}(\Omega))$$

is a Fourier multiplier. Since

$$\|M(t)\|_{\mathcal{L}(L^q_w(\Omega),Y_w^{\beta,q}(\Omega))} \leq \|\langle t\rangle^{-\frac{\beta}{2}+1}(it + \mathcal{A})^{-1}P_{q,w}\|_{\mathcal{L}(L^q_w(\Omega),Y_w^{\beta,q}(\Omega))}$$

for every t, we find by Lemma 9.4.3 that $M(t)$ is uniformly bounded by a constant that depends A_q-consently on w. By Theorem 9.4.2 this implies that $M(t)$ is R-bounded.

Moreover,

$$tM'(t) = (t\langle t\rangle^{-\frac{\beta}{2}} - \frac{\beta}{2}t^3\langle t\rangle^{-\frac{\beta}{2}-2})(it + \mathcal{A})^{-1}P_{q,w} - it^2\langle t\rangle^{-\frac{\beta}{2}}(it + \mathcal{A})^{-2}P_{q,w}.$$

Since $t(it+\mathcal{A})^{-1} : L^q_w(\Omega) \to L^q_w(\Omega)$ is uniformly bounded with an A_q-consistent constant, the second summand is R-bounded as before. Furthermore,

$$\left\|(t\langle t\rangle^{-\frac{\beta}{2}} - \frac{\beta}{2}t^3\langle t\rangle^{-\frac{\beta}{2}-2})(it+\mathcal{A})^{-1}P_{q,w}\right\|_{\mathcal{L}(L^q_w(\Omega),Y^{\beta,q}_w(\Omega))}$$
$$\leq \left\|(1+\frac{\beta}{2})\langle t\rangle^{1-\frac{\beta}{2}}(it+\mathcal{A})^{-1}P_{q,w}\right\|_{\mathcal{L}(L^q_w(\Omega),Y^{\beta,q}_w(\Omega))}$$

for every t. Thus we obtain the required R-boundedness of the first summand by an application of Theorem 9.4.2.

Combining the above with Theorem 2.4.5 and Lemma 9.4.1 shows that $M(t)$ is a multiplier. Thus

$$\|u_2\|_{L^r(H^{\beta,q}_w)} \leq \|u_2\|_{L^r(Y^{\beta,q}_w)} = \|\mathcal{F}^{-1}iM(t)\langle t\rangle^{\frac{\beta}{2}}\hat{u}_1\|_{L^r(H^{\beta,q}_w)}$$
$$\leq c\|\mathcal{F}^{-1}\langle t\rangle^{\frac{\beta}{2}}\hat{u}_1\|_{L^r(L^q_w)} = c\|u_1\|_{H^{\frac{\beta}{2},r}(L^q_w)} \leq c\|g\|_{H^{\frac{\beta}{2},r}(T^{0,q}_w)}.$$

Using this we are in the position to estimate the time derivative of u because

$$\partial_t u = \partial_t u_1 + \partial_t u_2 = \partial_t u_1 - \mathcal{A}u_2 - \partial_t u_1 = -\mathcal{A}u_2$$

yields

$$\|\partial_t u\|_{L^r(Y^{\beta-2,q}_{w,\sigma})} = \|\mathcal{A}u_2\|_{L^r(Y^{\beta-2,q}_{w,\sigma})} \leq c\|u_2\|_{L^r(Y^{\beta,q}_{w,\sigma})} \leq c\|g\|_{H^{\frac{\beta}{2},r}(T^{0,q}_w)}.$$

Combining this with the estimate for u_1 implies

$$\|u_t\|_{L^r(Y^{\beta-2,q}_{w,\sigma})} + \|u\|_{L^r(H^{\beta,q}_w)} \leq c\left(\|g\|_{L^r(T^{\beta,q}_w)} + \|g\|_{H^{\frac{\beta}{2},r}(T^{0,q}_w)}\right)$$
$$= c\|g\|_{L^r(T^{\beta,q}_w)\cap H^{\frac{\beta}{2},r}(T^{0,q}_w)}.$$

□

9.5 Solutions to Inhomogeneous Data in Bessel Potential Spaces

Whereas in the previous section the only inhomogeneous data was the part of the boundary condition which is tangential to the boundary, we are now dealing with inhomogeneous forces, divergences, initial conditions and tangential and normal part of the boundary condition. Our aim is to prove higher regularity of the solution in weighted Bessel potential spaces. We obtain the solution to an inhomogeneous force by interpolation between the very weak and the strong solution. The initial condition is represented by the semigroup generated by the generalized Stokes operator. The divergence and the normal part of the boundary condition can be realized by a gradient and the tangential part of the boundary condition has been treated in Section 9.4.

In the end, it remains to put all these parts together to obtain a solution with the asserted regularity. Thus this section combines many results and tools from the previous sections.

In the following we consider exterior forces

$$f \in L^r(0,T;Y_w^{\beta-2,q}(\Omega)) = \big[L^r(0,T;Y_w^{-2,q}(\Omega)), L^r(0,T;L_w^q(\Omega))\big]_{\frac{\beta}{2}} \quad \text{for} \quad 0 \le \beta \le 2,$$

where the equality of the spaces follows from [52, 1.18.4] combined with Theorem 7.4.2. For such forces one obtains very weak solutions to the instationary Stokes problem by interpolation.

To make use of the generalized Stokes operator in $Y_{w,\sigma}^{\beta-2,q}(\Omega)$ we restrict such forces f to test functions $\phi \in Y_{w',\sigma}^{2,q'}(\Omega)$.

Lemma 9.5.1. *For every $f \in L^r(0,T;Y_w^{\beta-2,q}(\Omega))$ there exists a unique solution $u \in L^r(0,T;Y_{w,\sigma}^{\beta,q}(\Omega))$ to the Stokes equation*

$$u_t + \mathcal{A}u = f|_{Y_{w',\sigma}^{2,q'}(\Omega)} \quad \textit{in} \quad \mathcal{D}'(0,T;Y_{w,\sigma}^{\beta-2,q}(\Omega)) \quad \textit{with} \quad u(0)|_{Y_{w',\sigma}^{2,q'}(\Omega)} = 0.$$

It fulfills the estimate

$$\|u\|_{L^r(Y_{w,\sigma}^{\beta,q})} \le c\|f|_{Y_{w',\sigma}^{2,q'}(\Omega)}\|_{L^r(Y_{w,\sigma}^{\beta-2,q})}.$$

Proof. By Corollary 9.2.2 and Lemma 9.2.4 this is true for $\beta = 0$. For $\beta = 2$ and $\phi \in Y_{w',\sigma}^{2,q'}(\Omega)$ one has

$$\langle f(t), \phi\rangle_\Omega = \langle f(t), P_{q',w'}\phi\rangle_\Omega = \langle P_{q,w}f(t), \phi\rangle_\Omega$$

showing $f|_{Y_{w',\sigma}^{2,q'}(\Omega)} = P_{q,w}f|_{Y_{w',\sigma}^{2,q'}(\Omega)}$. This means the solution operator

$$L : L^r(0,T;L_w^q(\Omega)) \ni f \mapsto u \in L^r(0,T;Y_{w,\sigma}^{2,q}(\Omega)),$$

where u is the strong solution to the instationary Stokes equations, is well-defined and continuous by Theorem 9.1.2. It coincides with the very weak solution with respect to $\phi \mapsto \langle f, \phi\rangle$ by the uniqueness of the very weak solution in Theorem 9.2.1.

Thus we may apply interpolation to the solution operator $L : f \mapsto u$

$$\begin{aligned} &L : L^r(0,T;Y_w^{-2,q}(\Omega)) \to L^r(0,T;L_{w,\sigma}^q(\Omega)) \quad \text{and} \\ &L : L^r(0,T;L_w^q(\Omega)) \to L^r(0,T;Y_{w,\sigma}^{2,q}(\Omega)) \end{aligned}$$

and we obtain the assertion. □

We assume the divergence k to be contained in

$$L^r(0,T;H_{w,*}^{\beta-1,q}(\Omega)) \cap H_0^{\frac{\beta}{2},r}((0,T];W_{w,0}^{-1,q}(\Omega)), \quad 0 \le \beta \le 2,$$

where, as in Section 8.1 we denote

$$H_{w,*}^{\beta,q}(\Omega) = \begin{cases} H_{w,0}^{\beta,q}(\Omega) = (H_{w'}^{-\beta,q'}(\Omega))' & \text{if} \quad \beta < 0, \\ H_w^{\beta,q}(\Omega) & \text{if} \quad \beta \ge 0. \end{cases}$$

Recall that in Theorem 7.3.4 we have shown the interpolation property

$$\big[W_{w,0}^{-1,q}(\Omega), W_w^{1,q}(\Omega)\big]_{\frac{1+\beta}{2}} = H_{w,*}^{\beta,q}(\Omega) \quad \text{for} \quad -1 \le \beta \le 1.$$

As a space of boundary values we consider

$$L^r(0,T;T_w^{\beta,q}(\partial\Omega)) \cap H_0^{\frac{\beta}{2},r}((0,T];T_w^{0,q}(\partial\Omega)).$$

However, one has to keep in mind that if $0 \le \beta < 1$ it is not clear whether the solution $u \in L^r(0,T;H_w^{\beta,q}(\Omega))$ is regular enough to ensure that the expression $u|_{\partial\Omega} = g$ is well-defined. Further discussions about boundary values in the case of sufficiently regular data and solutions can be found at the end of this section.

Our space of initial values is

$$\mathcal{I}_w^{\beta,q,r} = \mathcal{I}_w^{\beta,q,r}(\Omega) := \left\{ u_0 \in Y_{w,\sigma}^{\beta-2,q}(\Omega) \;\Big|\; \int_0^\infty \|e^{-t\mathcal{A}}u_0\|_{\beta,q,w}^r dt < \infty \right\},$$

where $e^{-t\mathcal{A}}$ is the semigroup that is generated by the generalized Stokes operator $\mathcal{A}$ in $Y_{w,\sigma}^{\beta-2,q}(\Omega)$ with

$$e^{-t\mathcal{A}} : Y_{w,\sigma}^{\beta-2,q}(\Omega) \to \mathcal{D}(\mathcal{A}) = Y_{w,\sigma}^{\beta,q}(\Omega) \subset H_{w,\sigma}^{\beta,q}(\Omega).$$

It is equipped with the norm

$$\|u_0\|_{\mathcal{I}_w^{\beta,q,r}} := \|u_0\|_{Y_{w,\sigma}^{\beta-2,q}} + \|e^{-t\mathcal{A}}u_0\|_{L^r(H_w^{\beta,q})}.$$

Lemma 9.5.2. *$\mathcal{I}_w^{2,q,r}$ is dense in $\mathcal{I}_w^{\beta,q,r}$ for every $\beta \in [0,2]$.*

Proof. If $\beta = 2$ nothing is to show. Thus we assume $\beta \in [0,2)$.

For $u_0 \in \mathcal{I}_w^{\beta,q,r}$ and $\lambda > 0$ we set

$$u_\lambda := \lambda(\lambda + \mathcal{A})^{-1}u_0.$$

Recall the inequalities

$$\|(\lambda+\mathcal{A})^{-1}x\|_{L_{w,\sigma}^q} \le c\|x\|_{Y_{w,\sigma}^{-2,q}} \quad \text{and} \quad \|(\lambda+\mathcal{A})^{-1}x\|_{Y_{w,\sigma}^{2,q}} \le c\|x\|_{L_{w,\sigma}^q},$$

which are true with c independent of λ by Theorem 6.1.4. Hence by Theorem 6.2.2 one has

$$\begin{aligned}
\|u_\lambda\|_{\mathcal{I}_w^{2,q,r}} &= \|u_\lambda\|_{q,w,\sigma} + \left(\int_0^\infty \|e^{-t\mathcal{A}}\lambda(\lambda+\mathcal{A})^{-1}u_0\|_{2,q,w}^r dt \right)^{\frac{1}{r}} \\
&\le c(\lambda)\|u_0\|_{Y_{w,\sigma}^{-2,q}} + \left(\int_0^\infty c(\lambda)\|e^{-t\mathcal{A}}u_0\|_{q,w}^r dt \right)^{\frac{1}{r}} \\
&\le c(\lambda)\|u_0\|_{\mathcal{I}_w^{\beta,q,r}}.
\end{aligned}$$

This yields $u_\lambda \in \mathcal{I}_w^{2,q,r}$. Moreover, one has

$$x(t) := e^{-t\mathcal{A}}u_0 \in \bigcap_{n\in\mathbb{N}} \mathcal{D}(\mathcal{A}^n) \subset Y_{w,\sigma}^{2,q}(\Omega)$$

and we find by Lemma 9.4.3

$$\begin{aligned}
\|\lambda(\lambda+\mathcal{A})^{-1}x(t) - x(t)\|_{Y_{w,\sigma}^{\beta,q}} &= \|\mathcal{A}(\lambda+\mathcal{A})^{-1}x(t)\|_{Y_{w,\sigma}^{\beta,q}} \\
&\le \frac{1}{\langle\lambda\rangle^{1-\frac{\beta}{2}}}\|\mathcal{A}x(t)\|_{q,w} \qquad (9.5.1)\\
&\xrightarrow{\lambda\to\infty} 0.
\end{aligned}$$

In addition by Lemma 8.2.3 one has the estimate

$$\|\mathcal{A}(\lambda+\mathcal{A})^{-1}x(t)\|_{Y^{\beta,q}_{w,\sigma}} \le c\|\mathcal{A}x(t)\|_{Y^{\beta-2,q}_{w,\sigma}} \in L^r(\mathbb{R}_+)$$

with c independent of λ. Thus by Lebesgue's Theorem we have

$$\|e^{-t\mathcal{A}}u_\lambda - e^{-t\mathcal{A}}u_0\|_{Y^{\beta,q}_{w,\sigma}} = \|\lambda(\lambda+\mathcal{A})^{-1}x(t) - x(t)\|_{Y^{\beta,q}_{w,\sigma}} \to 0 \ \text{ in } \ L^r(\mathbb{R}_+)$$

as $\lambda \to \infty$. In addition Lemma [40, Lemma I.3.2] implies that $u_\lambda \to u_0$ in $Y^{\beta-2,q}_{w,\sigma}(\Omega)$as $\lambda \to \infty$ and we obtain convergence in $\mathcal{I}^{\beta,q,r}_w$. □

Lemma 9.5.3. *Let $1<q<\infty$, $\beta \in [0,2]$ and let $w \in A_q$. Let $\Omega \subset \mathbb{R}^n$ be a bounded $C^{2,1}$-domain if $\beta > 1$ and a bounded $C^{1,1}$-domain if $\beta \le 1$.*

Then the Helmholtz projection

$$P_{q,w} : H^{\beta,q}_w(\Omega) \to H^{\beta,q}_w(\Omega)$$

is continuous.

Proof. This follows by interpolation of the corresponding assertions for $\beta = 0,1,2$. The assertion for $\beta = 0$ follows from Section 4.4 and the one for $\beta = 1$ and $\beta = 2$ has been stated in Theorem 4.4.1. □

Theorem 9.5.4. *Let $1<q<\infty$, $\beta \in [0,2]$ and let $w \in A_q$. Let $\Omega \subset \mathbb{R}^n$ be a bounded $C^{2,1}$-domain if $\beta > 1$ and a bounded $C^{1,1}$-domain if $\beta \le 1$. Moreover, we take*

$$\begin{aligned} f &\in L^r(0,T;Y^{\beta-2,q}_w(\Omega)),\\ k &\in H^{\frac{\beta}{2},r}_0((0,T];W^{-1,q}_{w,0}(\Omega)) \cap L^r(0,T;H^{\beta-1,q}_{w,*}(\Omega)),\\ g &\in H^{\frac{\beta}{2},r}_0((0,T];T^{0,q}_w(\partial\Omega)) \cap L^r(0,T;T^{\beta,q}_w(\partial\Omega)),\\ u_0 &\in \mathcal{I}^{\beta,q,r}_w(\Omega), \end{aligned}$$

fulfilling the compatibility condition

$$\langle k(t),1\rangle_\Omega = \langle g(t),N\rangle_{\partial\Omega}, \quad \text{for almost all} \ \ t \in (0,T).$$

Then there exists a unique very weak solution $u \in L^r(0,T;H^{\beta,q}_w(\Omega))$ to the instationary Stokes system, i.e.,

$$\begin{aligned} -\langle u,\phi_t\rangle_{\Omega,T} - \langle u,\Delta\phi\rangle_{\Omega,T} &= -\langle u_0,\phi(0)\rangle_\Omega + \langle f,\phi\rangle_{\Omega,T} - \langle g, N\cdot\nabla\phi\rangle_{\partial\Omega,T}\\ -\langle u(t),\nabla\psi\rangle_\Omega &= \langle k(t),\psi\rangle_\Omega - \langle g(t),N\psi\rangle_{\partial\Omega} \quad \text{for a.e.} \ \ t \in [0,T] \end{aligned}$$

for all $\phi \in X^{r',q'}_{w',\sigma}$ and $\psi \in W^{1,q'}_{w'}(\Omega)$.

Moreover, there exists a pressure functional $p \in H^{-1,r}(0,T;H^{\beta-1,q}_w(\Omega))$ that is unique modulo constants, such that

$$\partial_t u - \Delta u + \nabla p = f|_{C_0^\infty(\Omega)}$$

is fulfilled in the sense of distributions on $(0,T)\times\Omega$.

This solution (u,p) fulfills the estimate

$$\begin{aligned}\|u_t|_{Y^{2,q'}_{w',\sigma}(\Omega)}\|_{L^r(0,T;Y^{\beta-2,q}_{w,\sigma}(\Omega))} + \|u\|_{L^r(H^{\beta,q}_w)} + \|p\|_{H^{-1,r}(H^{\beta-1,q}_w)}\\ \leq c\Bigg(\|f\|_{L^r(H^{\beta-2,q}_w)} + \|k\|_{H_0^{\frac{\beta}{2},r}((0,T];W^{-1,q}_{w,0})\cap L^r(H^{\beta-1,q}_{w,*})}\\ + \|g\|_{H_0^{\frac{\beta}{2},r}((0,T];T^{0,q}_w)\cap L^r(T^{\beta,q}_w)} + \|u_0\|_{\mathcal{I}^{\beta,q,r}_w}\Bigg)\end{aligned} \tag{9.5.2}$$

with $c = c(\Omega, r, \beta, q, w) > 0$.

Remark 9.5.5. The right hand side in the above theorem is

$$[\phi \mapsto -\langle u_0, \phi(0)\rangle_\Omega + \langle f, \phi\rangle_{\Omega,T} - \langle g, N\cdot\nabla\phi\rangle_{\partial\Omega,T}] \in (X^{r',q'}_{w'})'.$$

This means the case of non-zero initial conditions requires no generalization of the definition of the very weak solution given in Definition 9.1.1.

Proof. Step 1. We start with the divergence and the normal part of the boundary condition.

Let $\tilde{u}_1(t) \in H^{\beta,q}_w(\Omega)$ be the very weak solution to the stationary Stokes system with external force 0, boundary condition $g(t)$ and divergence $k(t)$. Moreover, set $u_1(t) := \tilde{u}_1(t) - P_{q,w}\tilde{u}_1(t)$. Then one has by Lemma 9.5.3

$$u_1(t) \in H^{\beta,q}_w(\Omega), \quad u_1(t) = \nabla\pi(t)$$

and for almost every $t \in [0,T]$ and every $\psi \in W^{1,q'}_{w'}(\Omega)$ one has by Section 4.4

$$\langle\nabla\pi, \nabla\psi\rangle_\Omega = \langle u_1(t), \nabla\psi\rangle_\Omega = \langle\tilde{u}_1(t), \nabla\psi\rangle_\Omega = -\langle k(t), \psi\rangle_\Omega + \langle g(t), N\psi\rangle_{\partial\Omega}.$$

This function π can be chosen such that $\int_\Omega \pi = 0$.

The a priori estimate of the solution to the stationary problem combined with the continuity of $P_{q,w}$ on $H^{\beta,q}_w(\Omega)$ shown in Lemma 9.5.3 implies $u_1 \in L^r(0,T;H^{\beta,q}_w(\Omega))$. Thus by Lemma 9.3.2 one has $\partial_t u_1 \in H^{-1,r}(0,T;H^{\beta,q}_w(\Omega))$ and it cannot be expected to be a function in time. However, since u_1 is a gradient, for $\phi \in C_0^\infty(0,T;Y^{2,q'}_{w',\sigma}(\Omega))$ one has

$$\langle\partial_t u_1, \phi\rangle_{\Omega,T} = -\langle u_1, \partial_t P_{q,w}\phi\rangle_{\Omega,T} = -\langle P_{q,w}u_1, \partial_t\phi\rangle_{\Omega,T} = 0.$$

Thus the estimate for $\partial_t u_1|_{Y^{2,q'}_{w',\sigma}(\Omega)} \in L^r(0,T;Y^{\beta-2,q}_{w,\sigma}(\Omega))$ is obvious.

Next we have to show that the tangential component of the boundary value $\gamma(u_1)$ of u_1 is well-defined in the sense of Theorem 6.3.6 and fulfills the estimate

$$\begin{aligned}\|\gamma(u_1)\|_{L^r(T^{\beta,q}_w)\cap H_0^{\frac{\beta}{2},r}(T^{0,q}_w)} &\leq c\|u_1\|_{L^r(H^{\beta,q}_w)\cap H_0^{\frac{\beta}{2},r}(L^q_w)}\\ &\leq c\left(\|k\|_{L^r(H^{\beta-1,q}_{w,*})\cap H_0^{\frac{\beta}{2},r}(W^{-1,q}_{w,0})} + \|g\|_{L^r(T^{\beta,q}_w)\cap H_0^{\frac{\beta}{2},r}(T^{0,q}_w)}\right).\end{aligned} \tag{9.5.3}$$

We begin proving the following pointwise inequality

$$\|\gamma(u_1(t))\|_{T^{\beta,q}_w} \leq c\|u_1(t)\|_{H^{\beta,q}_w} \leq c(\|k(t)\|_{H^{\beta-1,q}_{w,*}} + \|g(t)\|_{T^{\beta,q}_w}). \tag{9.5.4}$$

The second inequality follows from the a priori estimate of the stationary Stokes equation in the Theorems 8.1.3 and 8.1.4 combined with the continuity of $P_{q,w}$. Hence it remains to prove the first.

If $\beta \geq 1$ this follows from the continuity of the restriction

$$v \mapsto v|_{\partial\Omega} : H_w^{\beta,q}(\Omega) \to T_w^{\beta,q}(\partial\Omega).$$

Thus we assume $0 \leq \beta < 1$. Since $\Delta u_1(t) = \nabla \Delta \pi(t)$ one has $\Delta u_1(t)|_{C^\infty_{0,\sigma}(\Omega)} = 0$. This means $\gamma(u_1(t)) \in T_w^{0,q}(\partial\Omega)$ is well-defined by Theorem 6.3.6. Moreover, if $\beta = 0$, this means that the mapping

$$W_w^{1,q}(\Omega) \ni \pi \mapsto \gamma(\nabla\pi) \in T_w^{0,q}(\partial\Omega)$$

is continuous and, by the definition of $T_w^{1,q}(\Omega)$, it is also bounded as an operator

$$\gamma \circ \nabla : W_w^{2,q}(\Omega) \to T_w^{1,q}(\partial\Omega).$$

Hence by interpolation we obtain the continuity of $\gamma \circ \nabla : H_w^{\beta+1,q}(\Omega) \to T_w^{\beta,q}(\partial\Omega)$ and this implies the pointwise estimate (9.5.4) for almost every t, where one uses the Lemmas 8.1.7 and 7.2.5 to verify

$$\|\pi\|_{H_w^{\beta+1,q}} \leq c\left(\|\nabla\pi\|_{H_w^{\beta,q}} + \|\pi\|_{H_w^{\beta,q}}\right) \leq c\left(\|\nabla\pi\|_{H_w^{\beta,q}} + \|\nabla\pi\|_{H_w^{\beta-1,q}}\right) \leq c\|u_1\|_{H_w^{\beta,q}},$$

Since π has mean value 0. Thus we obtain

$$\|\gamma(u_1)\|_{L^r(T_w^{\beta,q})} \leq c\|u_1\|_{L^r(H_w^{\beta,q})} \leq c(\|k\|_{L^r(H_{w,*}^{\beta-1,q})} + \|g\|_{L^r(T_w^{\beta,q})}). \tag{9.5.5}$$

In particular (9.5.4) holds for β replaced by 0. Assume for a moment that k, g and u_1 are defined on $\mathbb{R} \times \Omega$ with $\operatorname{supp} k, \operatorname{supp} g \subset [0, \infty)$ in time. Obviously the operator Λ_t acting in time commutes with the continuous operator $(g(t), k(t)) \mapsto u_1(t)$ acting in space. Combining this with (9.5.4) implies

$$\begin{aligned}\|\gamma(u_1)\|_{H^{\frac{\beta}{2},r}(\mathbb{R};T_w^{0,q})} \leq& c\|u_1\|_{H^{\frac{\beta}{2},r}(\mathbb{R};L_w^q)} \\ \leq& c\left(\|k\|_{H^{\frac{\beta}{2},r}(\mathbb{R};H_{w,0}^{-1,q})} + \|g\|_{H^{\frac{\beta}{2},r}(\mathbb{R};T_w^{0,q})}\right).\end{aligned} \tag{9.5.6}$$

For g and k given as in the assumption of this theorem by Lemma 9.3.2 there exist extensions $Eg \in H^{\frac{\beta}{2},r}(\mathbb{R}; T_w^{0,q}(\partial\Omega))$ and $Ek \in H^{\frac{\beta}{2},r}(\mathbb{R}; H_{w,0}^{-1,q}(\Omega))$. The resulting u_1^E fulfills $\operatorname{supp} u_1^E \subset \operatorname{supp} Eg \cup \operatorname{supp} Ek$ in time. This yields

$$\begin{aligned}\|\gamma(u_1)\|_{H_0^{\frac{\beta}{2},r}((0,T];T_w^{0,q})} \leq& \|\gamma(u_1^E)\|_{H^{\frac{\beta}{2},r}(\mathbb{R};T_w^{0,q})} \\ \leq& c\|u_1^E\|_{H^{\frac{\beta}{2},r}(\mathbb{R};L_w^q)} \\ \leq& c\left(\|Ek\|_{H^{\frac{\beta}{2},r}(\mathbb{R};H_{w,0}^{-1,q})} + \|Eg\|_{H^{\frac{\beta}{2},r}(\mathbb{R};T_w^{0,q})}\right) \\ \leq& c\left(\|k\|_{H_0^{\frac{\beta}{2},r}((0,T];H_{w,0}^{-1,q})} + \|g\|_{H_0^{\frac{\beta}{2},r}((0,T];T_w^{0,q})}\right).\end{aligned} \tag{9.5.7}$$

Combining (9.5.5) and (9.5.7) implies that the tangential component of the boundary value of u_1 fulfills

$$\gamma(u_1) \in H_0^{\frac{\beta}{2},r}((0,T];T_w^{0,q}(\partial\Omega)) \cap L^r(0,T;T_w^{\beta,q}(\partial\Omega))$$

and the estimate (9.5.3).
Step 2. We consider the tangential component of the boundary condition.
Let $u_2 \in L^r(0,T;H_w^{\beta,q}(\Omega))$ be the solution to the instationary Stokes system with vanishing initial condition, exterior force, divergence and the purely tangential boundary condition

$$g_{tan} - \gamma(u_1) \in H_0^{\frac{\beta}{2},r}((0,T];T_w^{0,q}(\partial\Omega)) \cap L^r(0,T;T_w^{\beta,q}(\partial\Omega)),$$

where g_{tan} is the tangential component of g. Such a function u_2 exists by Lemma 9.4.4 and fulfills the estimate

$$\begin{aligned}\|(\partial_t u_2)|_{Y^{2,q'}_{w',\sigma}}\|_{L^r(Y^{\beta-2,q}_{w,\sigma})} &+ \|u_2\|_{L^r(H^{\beta,q}_w)} \\ &\leq c\left(\|g_{tan}\|_{H_0^{\frac{\beta}{2},r}(T_w^{0,q})\cap L^r(T_w^{\beta,q})} + \|\gamma(u_1)\|_{H_0^{\frac{\beta}{2},r}(T_w^{0,q})\cap L^r(T_w^{\beta,q})}\right) \\ &\leq c\left(\|k\|_{H_0^{\frac{\beta}{2},r}(H^{-1,q}_{w,0})} + \|g\|_{H_0^{\frac{\beta}{2},r}(T_w^{0,q})} + \|k\|_{L^r(H^{\beta-1,q}_{w,*})} + \|g\|_{L^r(T_w^{\beta,q})}\right),\end{aligned}$$

where in the last inequality we have used (9.5.3).
Step 3. The next step is to consider the initial values.
We set $u_3(t) = e^{-t\mathcal{A}}u_0$, where $e^{-t\mathcal{A}}$ is the semigroup generated by the generalized Stokes operator in $Y^{\beta-2,q}_{w,\sigma}(\Omega)$. Then u_3 is a solution to

$$\partial_t u_3 + \mathcal{A}u_3 = 0, \qquad u_3|_{Y^{2,q'}_{w',\sigma}(\Omega)}(0) = u_0.$$

By the definition of the space of initial values $\mathcal{I}_w^{\beta,q,r}$ it fulfills the estimate

$$\|u_3\|_{L^r(H^{\beta,q}_w)} \leq \|u_0\|_{\mathcal{I}_w^{\beta,q,r}}.$$

The estimate of the time derivative follows from the equation by

$$\left\|\partial_t u_3|_{Y^{2,q'}_{w',\sigma}(\Omega)}\right\|_{L^r(Y^{\beta-2,q}_{w,\sigma})} = \|-\mathcal{A}u_3\|_{L^r(Y^{\beta-2,q}_{w,\sigma})} \leq c\|u_3\|_{L^r(Y^{\beta,q}_{w,\sigma})} \leq c\|u_0\|_{\mathcal{I}_w^{\beta,q,r}}.$$

Step 4. It remains to treat the exterior force.
By Lemma 9.5.1 there exists a unique very weak solution $u_4 \in L^r(0,T;Y^{\beta,q}_{w,\sigma}(\Omega))$ solving

$$\partial_t u_4 + \mathcal{A}u_4 = f|_{Y^{2,q'}_{w',\sigma}(\Omega)}, \quad u_4|_{Y^{2,q'}_{w',\sigma}(\Omega)}(0) = 0.$$

It fulfills the estimate

$$\left\|\partial_t u_4|_{Y^{2,q'}_{w',\sigma}(\Omega)}\right\|_{L^r(Y^{\beta-2,q}_{w,\sigma})} + \|u_4\|_{L^r(Y^{\beta,q}_w(\Omega))} \leq c\|f\|_{L^r(Y^{\beta-2,q}_w)},$$

where the estimate of the time derivative follows from the equation as in Step 3.
Step 5. Summarizing the above shows that $u := u_1 + u_2 + u_3 + u_4 \in L^r(0,T; H_w^{\beta,q}(\Omega))$ is a very weak solution as required, since for every $\phi \in X_{w',\sigma}^{q',r'}$ one has

$$\begin{aligned}
-\langle u, \partial_t\phi\rangle_{\Omega,T} &- \langle u, \Delta\phi\rangle_{\Omega,T} \\
&= - \langle u_1, \Delta\phi\rangle_{\Omega,T} - \langle u_2, \partial_t\phi\rangle_{\Omega,T} - \langle u_2, \Delta\phi\rangle_{\Omega,T} - \langle u_3, \partial_t\phi\rangle_{\Omega,T} \\
&\quad - \langle u_3, \Delta\phi\rangle_{\Omega,T} - \langle u_4, \partial_t\phi\rangle_{\Omega,T} - \langle u_4, \Delta\phi\rangle_{\Omega,T} \\
&= - \langle \gamma(u_1), N\cdot\nabla\phi\rangle_{\partial\Omega,T} - \langle g-\gamma(u_1), N\cdot\nabla\phi\rangle_{\partial\Omega,T} - \langle u_0, \phi(0)\rangle_{\Omega} + \langle f, \phi\rangle_{\Omega,T} \\
&= - \langle g, N\cdot\nabla\phi\rangle_{\partial\Omega,T} - \langle u_0, \phi(0)\rangle_{\Omega} + \langle f, \phi\rangle_{\Omega,T}.
\end{aligned}$$

In the first equation we have used that with the notation of (6.3.2)

$$\langle \gamma(u_1), N\cdot\nabla\phi\rangle_{\partial\Omega,T} = -\langle \Delta_\sigma u_1, \phi\rangle_{\Omega,T} + \langle u_1, \Delta\phi\rangle_{\Omega,T} = \langle u_1, \Delta\phi\rangle_{\Omega,T},$$

since $\Delta_\sigma u_1 = 0$ for the gradient u_1.

The function u fulfills the estimate

$$\begin{aligned}
\left\| \partial_t u|_{Y_{w',\sigma}^{2,q'}(\Omega)} \right\|_{L^r(Y_{w,\sigma}^{\beta-2,q})} + \|u\|_{L^r(H_w^{\beta,q})} \leq & c \Bigg(\|f\|_{L^r(H_w^{\beta-2,q})} + \|k\|_{L^r(H_{w,*}^{\beta-1,q}) \cap H_0^{\frac{\beta}{2},r}(W_{w,0}^{-1,q})} \\
& + \|g\|_{L^r(T_w^{\beta,q}) \cap H_0^{\frac{\beta}{2},r}(T_w^{0,q})} + \|u_0\|_{\mathcal{I}_w^{\beta,q,r}} \Bigg),
\end{aligned}$$

since this is true for u_1, u_2, u_3 and u_4.
Step 6. Let U be another very weak solution to the instationary Stokes system with respect to the same data. Then $U-u$ fulfills

$$\begin{aligned}
-\langle U-u, \phi_t\rangle_{\Omega,T} - \langle U-u, \Delta\phi\rangle_{\Omega,T} = 0 &\quad \text{for every} \quad \phi \in X_{w',\sigma}^{r',q'} \quad \text{and} \\
-\langle U(t)-u(t), \nabla\psi\rangle_{\Omega} = 0 &\quad \text{for every} \quad \psi \in W_{w'}^{1,q'}(\Omega) \\
&\quad \text{and almost every} \quad t.
\end{aligned}$$

By the uniqueness of the very weak solution proved in Theorem 9.2.1 we obtain $u = U$.
Step 7. It remains to show existence and estimates for the pressure functional.

We approximate f, k, g, u_0 by functions

$$\begin{aligned}
f_n &\in L^r(0,T; L_w^q(\Omega)), & k_n &\in H_0^{1,r}(0,T; H_w^{1,q}(\Omega)), \\
g_n &\in H_0^{1,r}(0,T; T_w^{2,q}(\partial\Omega)), & u_{0,n} &\in \mathcal{I}_w^{2,q,r}
\end{aligned}$$

in the norms of the corresponding spaces for the data as in the assumptions of this theorem.

Then one obtains as above a strong solution

$$u_n \in L^r(0,T; H_w^{2,q}(\Omega)) \quad \text{with} \quad \partial_t u_n \in L^r(0,T; L_w^q(\Omega))$$

to the Stokes problem with respect to the data $u_{0,n}, f_n, g_n, k_n$. By the uniqueness proved in Step 5 the functions u_n fulfill the a priori estimate (9.5.2). This implies $u_n \to u$ in $L^r(0,T; H_w^{\beta,q}(\Omega))$.

Let $\psi = \eta\phi$, where $\eta \in C_0^\infty(0,T)$ and $\phi \in C_{0,\sigma}^\infty(\Omega)$. Then one has

$$\int_0^T \langle \partial_t u_n, \phi\rangle_\Omega \eta\, dt - \int_0^T \langle u_n, \Delta\phi\rangle_\Omega \eta\, dt - \int_0^T \langle f_n, \phi\rangle_\Omega \eta\, dt = 0.$$

Since η was chosen arbitrarily, we find

$$\langle \partial_t u_n(t), \phi\rangle_\Omega - \langle u_n(t), \Delta\phi\rangle_\Omega - \langle f_n(t), \phi\rangle_\Omega = 0 \quad \text{for every} \quad \phi \in C_{0,\sigma}^\infty(\Omega)$$

and almost every t. By de Rham's Theorem [51] there exists $p_n(t) \in (C_0^\infty(\Omega))'$ such that

$$\partial_t u_n(t) - \Delta u_n(t) + \nabla p_n(t) = f_n(t) \quad \text{almost everywhere on} \quad (0,T)\times\Omega.$$

Since $\nabla p_n \in L^r(0,T;L_w^q(\Omega))$ one has by Lemma 8.1.7 that $p(t) \in W_w^{1,q}(\Omega)$ for almost every t. We choose $p_n(t)$ such that $\int p_n(t)\,dx = 0$ for every t.

Every ∇p_n fulfills the estimate

$$\begin{aligned}\|\nabla p_n\|_{H^{-1,r}(H_w^{\beta-2,q})} \le& \|\Delta u_n\|_{H^{-1,r}(H_w^{\beta-2,q})} + \|\partial_t u_n\|_{H^{-1,r}(H_w^{\beta-2,q})} + \|f_n\|_{H^{-1,r}(H_w^{\beta-2,q})}\\ \le& c\big(\|k_n\|_{L^r(H_{w,*}^{\beta-1,q})\cap H_0^{\frac{\beta}{2},r}(H_{w,0}^{-1,q})} + \|g_n\|_{L^r(T_w^{\beta,q})\cap H_0^{\frac{\beta}{2},r}(T_w^{0,q})}\\ &+ \|f_n\|_{L^r(Y_w^{\beta-2,q})} + \|u_{0,n}\|_{\mathcal{I}_w^{\beta,q,r}}\big),\end{aligned}$$

where we have used $Y_w^{\beta-2,q}(\Omega)|_{H_{w',0}^{2-\beta,q'}} \hookrightarrow H_w^{\beta-2,q}(\Omega)$ and Lemma 9.3.2 to show

$$\|\partial_t u_n\|_{H^{-1,r}(H_w^{\beta-2,q})} \le c\|u_n\|_{L^r(H_w^{\beta-2,q})} \le c\|u_n\|_{L^r(H_w^{\beta,q})}.$$

Moreover, by Lemma 9.3.2 one has $H^{-1,r}(0,T;H_w^{\beta-2,q}(\Omega)) = W^{-1,r}(0,T;H_w^{\beta-2,q}(\Omega))$ and as in the proof of Lemma 8.1.7 for every $\phi \in W_0^{1,r}(0,T;H_{w',0}^{1-\beta,q'}(\Omega))$ with mean value 0 we find $\zeta \in W_0^{1,r}(0,T;H_{w',0}^{2-\beta,q'}(\Omega))$ with

$$-\langle \zeta, \nabla\psi\rangle_\Omega = \langle \phi, \psi\rangle_\Omega \quad \text{for all} \quad \psi \in W_{w'}^{1,q'}(\Omega)$$

and $\|\zeta\|_{W_0^{1,r}(H_{w',0}^{2-\beta,q'})} \le c\|\phi\|_{W_0^{1,r}(H_{w',0}^{1-\beta,q'})}$. Thus for $\phi \in C_0^\infty((0,T)\times\Omega)$ with mean value 0 one has the estimate

$$\begin{aligned}|\langle p_n, \phi\rangle_{\Omega,T}| &= |\langle \nabla p_n, \zeta\rangle_{\Omega,T}|\\ &\le c\|\nabla p_n\|_{H^{-1,r}(H_w^{\beta-2,q})}\|\zeta\|_{W_0^{1,r}(H_{w',0}^{2-\beta,q'})}\\ &\le c\|\nabla p_n\|_{H^{-1,r}(H_w^{\beta-2,q})}\|\phi\|_{H_0^{1,r}(H_{w',0}^{1-\beta,q'})}.\end{aligned}$$

Combining the above yields the estimate

$$\begin{aligned}\|p_n\|_{H^{-1,r}(H_w^{\beta-1,q})} \le& c\big(\|k_n\|_{L^r(H_{w,*}^{\beta-1,q})\cap H^{\frac{\beta}{2},r}(H_{w,0}^{-1,q})} + \|g_n\|_{L^r(T_w^{\beta,q})\cap H_0^{\frac{\beta}{2},r}(T_w^{0,q})}\\ &+ \|f_n\|_{L^r(H_w^{\beta-2,q})} + \|u_{0,n}\|_{\mathcal{I}_w^{\beta,q,r}}\big).\end{aligned}$$

Replacing p_n by $p_n - p_m$ in the above estimates shows that (p_n) is a Cauchy sequence in $H^{-1,r}(0,T;H_w^{\beta-1,q}(\Omega))$ converging to some $p \in H^{-1,r}(0,T;H_w^{\beta-1,q}(\Omega))$.

The couple (u,p) solves the Stokes equations in the distributional sense and fulfills the a priori estimate. □

Note that the solution constructed in Theorem 9.5.4 does in general not fulfill

$$\partial_t u \in L^r(0,T;Y_w^{\beta-2,q}(\Omega)).$$

The part which is irregular in time is given by the gradient u_1, which was needed for the nonhomogeneous divergence and normal component of the boundary condition. It vanishes when restricting $\partial_t u$ to $\phi \in Y_{w',\sigma}^{2,q'}(\Omega)$.

The same fact is the reason for the pressure p to be contained only in the space $H^{-1,r}(0,T;H_w^{\beta-1,q}(\Omega))$. This result could be improved to

$$p \in H^{-1,r}(0,T;H_w^{\beta,q}(\Omega)) + L^r(0,T;H_w^{\beta-1,q}(\Omega)),$$

but p is in general not integrable in time.

Another problem concerns the boundary values. In the above theorem boundary conditions are included even though for $0 \le \beta < 1$ the equation $u|_{\partial\Omega} = g$ in general makes no sense. The reason is that u is in general not smooth enough to make its restriction to the boundary well-defined.

However, if data and solution are regular enough, this can be established a posteriory. More precisely, let $\mu \in (1,\infty)$ and $\tilde{w} \in A_\mu$ such that

$$L_{\tilde{w}}^\mu(\Omega) \hookrightarrow W_{w,0}^{\beta-1,q}(\Omega)$$

and assume $k \in L^r(0,T;L_{\tilde{w}}^\mu(\Omega)) \cap H_0^{\frac{\beta}{2},r}((0,T];H_{w,0}^{-1,q}(\Omega))$. Then the normal component of the boundary condition can be defined as in the stationary case and one obtains

$$\begin{aligned}
\langle u(t), N\psi\rangle_{\partial\Omega} &= \langle u(t), \nabla\psi\rangle_\Omega + \langle \operatorname{div} u(t), \psi\rangle_\Omega \\
&= -\langle k(t), \psi\rangle_\Omega + \langle g(t), N\psi\rangle_{\partial\Omega} + \langle k(t), \psi\rangle_\Omega \\
&= \langle g(t), N\psi\rangle_{\partial\Omega}
\end{aligned}$$

for almost every t and every $\psi \in W_{w'}^{1,q'}(\Omega)$. Thus the normal component of u is equal to the one of g.

The tangential component causes more difficulties than in the stationary case. The reason is that $f \in L^r(0,T;W_{\tilde{w}}^{-1,\mu}(\Omega))$ does in general not imply $\partial_t u(t) \in W_{\tilde{w}}^{-1,\mu}(\Omega)$ for almost every t. And this is necessary to ensure $u(t) \in \tilde{W}_{w,\tilde{w}}^{q,\mu}$, the space in which the tangential component of the boundary values is well-defined.

Hence, to ensure that the tangential boundary condition is well-defined we assume

$$f \in L^r(0,T;W_{\tilde{w}}^{-1,\mu}(\Omega)) \quad \text{and} \quad u \in L^r(0,T;H_w^{\beta,q}(\Omega)), \;\; u_t(t) \in W_{\tilde{w}}^{-1,\mu}(\Omega) \tag{9.5.8}$$

for almost every t. Then, using test functions as in Step 7 of the proof of Theorem 9.5.4 one shows that for every $\phi \in C_{0,\sigma}^\infty(\Omega)$ and almost every t one has

$$\langle \Delta u(t), \phi\rangle_\Omega = \langle u(t), \Delta\phi\rangle_\Omega = \langle \partial_t u(t), \phi\rangle_\Omega - \langle f(t), \phi\rangle_\Omega$$

which implies $u(t) \in \tilde{W}_{w,\tilde{w}}^{q,\mu}$ for almost every t by the assumptions on f and u_t. Moreover,

$$\begin{aligned}
\langle u, N\cdot\nabla\phi\rangle_{\partial\Omega,T} &= \langle u, \Delta\phi\rangle_{\Omega,T} - \langle \Delta u, \phi\rangle_{\Omega,T} \\
&= -\langle u, \partial_t\phi\rangle_{\Omega,T} - \langle f, \phi\rangle_{\Omega,T} + \langle g, N\cdot\nabla\phi\rangle_{\partial\Omega,T} - \langle \partial_t u, \phi\rangle_{\Omega,T} + \langle f, \phi\rangle_{\Omega,T} \\
&= \langle g, N\cdot\nabla\phi\rangle_{\partial\Omega,T}
\end{aligned}$$

for every $\phi \in W^{1,r}(0,T;Y^{2,q'}_{w',\sigma}(\Omega))$ with $\phi(0) = \phi(T) = 0$. This means that u fulfills the tangential boundary condition almost everywhere.

In particular, (9.5.8) is fulfilled in the case of weak solutions. Thus one has the following proposition.

Proposition 9.5.6. *Let $\beta \in [1,2]$ and the data f, k, g and u_0 be chosen according to Theorem 9.5.4 and let $u \in L^r(0,T;H^{\beta,q}_w(\Omega))$ be a very weak solution with respect to this data.*

Then u, p fulfills the Stokes system (9.0.1) in the sense of distributions. In addition $u(t)|_{\partial\Omega} = g(t)$ for almost every t.

10 Stationary Navier-Stokes Equations with Irregular Data

The aim of this chapter is to find a solution theory to the stationary Navier-Stokes equations,

$$-\Delta u + u \cdot \nabla u + \nabla p = F \quad \text{in} \quad \Omega \tag{10.0.1}$$

$$\operatorname{div} u = K \quad \text{in} \quad \Omega \tag{10.0.2}$$

$$u|_{\partial\Omega} = g, \tag{10.0.3}$$

in a bounded domains $\Omega \subset \mathbb{R}^n$ with data F, K, g as general as possible. However, the nonlinearity gives us reason to demand higher regularity of data and solutions. First of all, the nonlinear term can be written as

$$u \cdot \nabla u = \operatorname{div} uu - Ku.$$

To make the multiplication on the right hand side well-defined, it is reasonable to demand that K is given by a function.

Moreover, when estimating the nonlinear term, one needs a weighted analogue to the Sobolev Embedding Theorem. A good replacement has been proved by Fröhlich [28] based on the continuity of singular integral operators shown by Sawyer and Wheeden [41]. These embedding theorems require strong assumptions to the weight function.

This can be compensated for the price of restrictions to the generality of the data and consequently of a smaller class of solution. Thus it is natural to consider the problem in Bessel potential spaces, where we are able to adapt the regularity of data and solutions precisely to the generality of the weight function.

10.1 Estimates of the Nonlinear Term

We prepare some embedding theorems. These theorems are proved by the use of weakly singular integral operators. Thus for $0 < \beta < n$ we define

$$I_\beta g(x) = \int_{\mathbb{R}^n} \frac{g(y)}{|x-y|^{n-\beta}} dy = c\mathcal{F}^{-1}|\xi|^{-\beta}\hat{g}(x), \tag{10.1.1}$$

where the second equality holds by [48, V. Lemma 2] for an appropriate constant $c \in \mathbb{R}$.

Theorem 10.1.1. *Let $0 < \beta < n$ and $1 < p < q < \infty$, $v \in A_p$ and $w \in A_q$. Moreover, assume that v and w fulfill the condition*

$$|Q|^{\frac{\beta}{n}-1} \left(\int_Q w \right)^{\frac{1}{q}} \left(\int_Q v^{-\frac{1}{p-1}} \right)^{\frac{1}{p'}} < c \quad \textit{for every cube} \quad Q \subset \mathbb{R}^n$$

with a constant $c > 0$ independent of Q. Then

$$\|I_\beta f\|_{q,w} \le c\|f\|_{p,v} \quad \text{for every} \quad f \in L^p_v(\mathbb{R}^n).$$

Proof. This is a special case of [41, Theorem 1 (B)]. □

Lemma 10.1.2. *Let $w \in A_q$, $v \in A_p$ with*

$$|Q|^{\frac{\beta}{n}-1}\left(\int_Q w\right)^{\frac{1}{q}}\left(\int_Q v^{-\frac{1}{p-1}}\right)^{\frac{1}{p'}} < c \quad \text{for every cube} \quad Q \subset \mathbb{R}^n$$

with a constant $c > 0$ independent of Q. Then one has

$$H^{\gamma,p}_v(\mathbb{R}^n) \hookrightarrow L^q_w(\mathbb{R}^n) \quad \text{for every} \quad \gamma \ge \beta.$$

Proof. By [28, Lemma 3.2] the embedding

$$\mathcal{M} := \left\{f \in \mathcal{S}(\mathbb{R}^n) \mid \hat{f} \equiv 0 \text{ in a neighborhood of } 0\right\} \hookrightarrow H^{\beta,p}_v(\mathbb{R}^n)$$

is dense. Moreover, we define

$$J_\beta f := c\mathcal{F}^{-1}|\xi|^\beta(1+|\xi|^2)^{-\frac{\beta}{2}}\mathcal{F}f,$$

where c is the constant from (10.1.1). Then by the Multiplier Theorem 3.2.4 the operator $J_\beta : L^p_v(\Omega) \to L^p_v(\Omega)$ is continuous. Moreover, for $f \in \mathcal{M}$ one has $f = I_\beta J_\beta \Lambda_\beta f$. Thus one obtains using Theorem 10.1.1 for every $f \in \mathcal{M}$

$$\|f\|_{L^q_w(\mathbb{R}^n)} = \|I_\beta J_\beta \Lambda_\beta f\|_{L^q_w(\mathbb{R}^n)} \le c\|J_\beta \Lambda_\beta f\|_{L^p_v(\mathbb{R}^n)} \le c\|\Lambda_\beta f\|_{L^p_v(\mathbb{R}^n)} = c\|f\|_{H^{\beta,p}_v(\mathbb{R}^n)}.$$

Thus by the density of $\mathcal{M}$ in $H^{\beta,p}_v(\mathbb{R}^n)$ the inequality holds for every $f \in H^{\beta,p}_v(\mathbb{R}^n)$ and one obtains

$$H^{\gamma,p}_v(\mathbb{R}^n) \hookrightarrow H^{\beta,p}_v(\mathbb{R}^n) \hookrightarrow L^q_w(\mathbb{R}^n).$$

□

Lemma 10.1.3. *Let $1 < q < \infty$, $w \in A_q$, $1 \le s < q$ and $\Omega \subset \mathbb{R}^n$ be bounded and open. Moreover, we assume that there exists $\delta > 0$ and a constant $c > 0$ such that*

$$|Q|^s \le cw(Q) \quad \text{for every cube} \quad Q \subset \Omega_\delta := \{x \in \mathbb{R}^n,\ \operatorname{dist}(x,\Omega) \le \delta\}.$$

Then there exists a weight function $W \in A_q$ with $w|_\Omega = W|_\Omega$ and

$$|Q|^s \le cW(Q) \quad \text{for every cube} \quad Q \subset \mathbb{R}^n.$$

Proof. [25, Lemma A.2] □

Lemma 10.1.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Moreover, let $1 \le s \le r \le q < \infty$, $r > 1$ and assume $0 \le \beta < n$ such that*

$$\frac{1}{q} \ge \frac{1}{r} - \frac{\beta}{ns}. \tag{10.1.2}$$

Then for every $w \in A_s$ the following embeddings are true:

1. $H_w^{\beta,r}(\Omega) \hookrightarrow L_w^q(\Omega)$.

2. $H_{w_q}^{\beta,q'}(\Omega) \hookrightarrow L_{w_r}^{r'}(\Omega)$, *where* $w_q = w^{-\frac{1}{q-1}}$ *and* $w_r = w^{-\frac{1}{r-1}}$.

3. $L_w^r(\Omega) \hookrightarrow H_w^{-\beta,q}(\Omega)$, $L_w^r(\Omega) \hookrightarrow H_{w,0}^{-\beta,q}(\Omega)$ *and for* $\beta \in [0,1]$ *one has* $W_w^{-1,r}(\Omega) \hookrightarrow Y_w^{-1-\beta,q}(\Omega)$.

4. *If* $\beta \in [0,1]$, *then one has* $H_w^{1,r}(\Omega) \hookrightarrow H_w^{1-\beta,q}(\Omega)$.

Proof. We begin with showing that without loss of generality we may assume that $1 \le s < r$.

Let $s = r$. Since $r > 1$ and $w \in A_r$ by Lemma 3.1.2.3 there exists $t \in [1,r)$ such that $w \in A_t$. If (10.1.2) holds for s, it holds for s replaced by t in any case. Thus we may replace s by $t < r$.

1. By [28, Corollary 3.2] the asserted embedding holds if there exists a constant $C > 0$ such that $|Q|^{\frac{\beta}{n}} w(Q)^{\frac{1}{q}-\frac{1}{r}} < C$ for all $Q \subset U$ for some open set $U \supset \overline{\Omega}$. By Lemma 3.1.2.1 we know that for every $Q \subset U$ and $w \in A_s$ it holds $|Q|^s \le \frac{|U|^s}{w(U)} w(Q) = cw(Q)$. Thus

$$|Q|^{\frac{\beta}{n}} w(Q)^{\frac{1}{q}-\frac{1}{r}} \le cw(Q)^{\frac{\beta}{sn}+\frac{1}{q}-\frac{1}{r}} \le cw(U)^{\frac{\beta}{sn}+\frac{1}{q}-\frac{1}{r}} =: C$$

since $\frac{\beta}{sn} + \frac{1}{q} - \frac{1}{r} \ge 0$ by assumption.

2. As above Lemma 3.1.2.1 states that $w \in A_s$ implies $w(Q) \ge c(U)|Q|^s$ for every $Q \subset U$, where U is some bounded domain with $\overline{\Omega} \subset U$. Thus by Lemma 10.1.3 there exists a weight function $W \in A_q$ such that $W = w$ on Ω and $W(Q) \ge c(U)|Q|^s$ for every cube $Q \subset \mathbb{R}^n$.

Now by Theorem 7.1.2 we know that

$$H_{w_q}^{\gamma,q'}(\Omega) = H_{W_q}^{\gamma,q'}(\Omega)$$

with equivalent norms. By Lemma 10.1.2 the condition

$$|Q|^{\frac{\alpha}{n}-1} \left(\int_Q W_r\right)^{\frac{1}{r'}} \left(\int_Q (W_q)^{-\frac{1}{q'-1}}\right)^{\frac{1}{q}} < c \;\text{ for every cube }\; Q \subset \mathbb{R}^n \tag{10.1.3}$$

implies

$$H_{W_q}^{\gamma,q'}(\mathbb{R}^n) \hookrightarrow L_{W_r}^{r'}(\mathbb{R}^n) \;\text{ for every }\; \gamma \ge \alpha.$$

Thus we have to show (10.1.3).

Since

$$W_r^{-\frac{1}{r'-1}} = W^{\frac{1}{r'-1}\frac{1}{r-1}} = W = (W_q)^{-\frac{1}{q'-1}},$$

we calculate using the definition of Muckenhoupt weights, $W \in A_r$ and $\frac{1}{q} - \frac{1}{r} \le 0$

$$\begin{aligned} |Q|^{\frac{\alpha}{n}-1} \left(\int_Q W_r\right)^{\frac{1}{r'}} \left(\int_Q (W_q)^{-\frac{1}{q'-1}}\right)^{\frac{1}{q}} &= |Q|^{\frac{\alpha}{n}-1} W_r(Q)^{\frac{1}{r'}} W(Q)^{\frac{1}{q}} \\ &\le c|Q|^{\frac{\alpha}{n}} W(Q)^{(\frac{1}{q}-\frac{1}{r})} \\ &\le c|Q|^{\frac{\alpha}{n}+s(\frac{1}{q}-\frac{1}{r})}. \end{aligned}$$

The last term is bounded if $\frac{\alpha}{n} + s(\frac{1}{q} - \frac{1}{r}) = 0$. There exists $0 \le \alpha \le \beta$ so that this is true, because $s\left(\frac{1}{q} - \frac{1}{r}\right) \le 0$ and for $\alpha = \beta$ one has $\frac{\beta}{n} + s(\frac{1}{q} - \frac{1}{r}) \ge \frac{\beta}{n} - s\frac{\beta}{sn} = 0$.

Now for $f \in H^{\gamma,q'}_{w_q}(\Omega)$ there exists an extension $F \in H^{\gamma,q'}_{W_q}(\mathbb{R}^n)$ with $\|F\|_{H^{\gamma,q'}_{W_q}(\mathbb{R}^n)} \le 2\|f\|_{H^{\gamma,q'}_{W_q}(\Omega)} \le c\|f\|_{H^{\gamma,q'}_{w_q}(\Omega)}$. One obtains

$$\|f\|_{L^{r'}_{w_r}(\Omega)} \le \|F\|_{L^{r'}_{W_r}(\mathbb{R}^n)} \le c\|F\|_{H^{\gamma,q'}_{W_q}(\mathbb{R}^n)} \le c\|f\|_{H^{\gamma,q'}_{W_q}(\mathbb{R}^n)},$$

and the asserted embedding is proved.

3. Considering the dual spaces in 2. we obtain

$$L^r_w(\Omega) \hookrightarrow H^{-\beta,q}_{w,0}(\Omega).$$

Moreover, since $H^{\beta,q'}_{w',0}(\Omega) \hookrightarrow H^{\beta,q'}_{w'}(\Omega)$, one has for $u \in L^r_w(\Omega)$ and every $\phi \in H^{\beta,q'}_{w',0}(\Omega)$

$$|\langle u, \phi\rangle| \le \|u\|_{H^{-\beta,q}_{w,0}(\Omega)}\|\phi\|_{H^{\beta,q'}_{w'}(\Omega)} \le c\|u\|_{L^r_w(\Omega)}\|\phi\|_{H^{\beta,q'}_{w',0}(\Omega)},$$

and this yields $L^r_w(\Omega) \hookrightarrow H^{-\beta,q}_w(\Omega)$. Finally, for $u \in W^{-1,r}_w(\Omega)$ and $\phi \in Y^{2,q'}_{w'}(\Omega)$ one has by the Poincaré inequality

$$\begin{aligned}|\langle u, \phi\rangle| &\le \|u\|_{-1,r,w}\|\phi\|_{1,r',w'} \le c\|u\|_{-1,r,w}\|\nabla\phi\|_{r',w'}\\ &\le c\|u\|_{-1,r,w}\|\nabla\phi\|_{\beta,q',w'} \le c\|u\|_{-1,r,w}\|\phi\|_{\beta+1,q',w'}.\end{aligned}$$

This proves the last embedding.

4. For $u \in H^{1,r}_w(\Omega)$ one has by Lemma 8.1.7 and 3.

$$\left\|u - \int_\Omega u\,dx\right\|_{1-\beta,q,w} \le c\|\nabla u\|_{-\beta,q,w} \le c\|\nabla u\|_{r,w} \le c\|u\|_{1,r,w}.$$

Thus

$$\|u\|_{1-\beta,q,w} \le c\|u\|_{1,r,w} + \int_\Omega |u|\,dx \le c\|u\|_{1,r,w} + c\|u\|_{r,w} \le c\|u\|_{1,r,w}.$$

□

Lemma 10.1.5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded $C^{1,1}$-domain. Assume $w \in A_s$ for some $1 \le s < q$ and $\beta > \frac{ns}{q} - 1$ in the case $n \ge 3$ and $\beta > \frac{2s}{q} - \frac{1}{2}$ in the case $n = 2$.*

1. In addition, let $0 \le \beta \le 1$ and $1 < t < \infty$ with

$$\frac{1-\beta}{ns} + \frac{1}{q} - \frac{1}{t} = 0. \tag{10.1.4}$$

Then $w \in A_t$,

$$L^t_w(\Omega) \hookrightarrow H^{\beta-1,q}_{w,0}(\Omega)$$

and

a)

$$\left|\int uv\psi\,dx\right| \le c\|u\|_{\beta,q,w}\|v\|_{\beta,q,w}\|\psi\|_{t',w'}$$

for every $u, v \in H^{\beta,q}_w(\Omega)$ and $\psi \in H^{1-\beta,q'}_{w'}(\Omega)$.

b)

$$\left|\int ku\phi\,dx\right| \le c\|k\|_{t,w}\|u\|_{\beta,q,w}\|\phi\|_{1,t',w'}$$

for every $k \in L^t_w(\Omega)$, $u \in H^{\beta,q}_w(\Omega)$ *and* $\phi \in H^{2-\beta,q'}_{w'}(\Omega)$.

2. *If* $1 \le \beta \le 2$ *then*

$$\|u\nabla v\|_{\beta-2,q,w} \le c\|u\|_{\beta,q,w}\|v\|_{\beta,q,w} \quad \text{for every} \quad u,v \in H^{\beta,q}_w(\Omega).$$

Proof. One has

$$t = \frac{nsq}{q(1-\beta)+ns} > \frac{nsq}{q(2-\frac{ns}{q})+ns} = \frac{ns}{2} \ge s.$$

Thus, by Lemma 10.1.4 one has $L^t_w(\Omega) \hookrightarrow H^{\beta-1,q}_{w,0}(\Omega)$ and $H^{1-\beta,q'}_{w_q}(\Omega) \hookrightarrow L^{t'}_{w_t}(\Omega)$.
1. a) Let $r := 2t$. Then one has

- $\frac{1}{r} - \frac{1}{q} + \frac{\beta}{ns} \ge 0$ and hence $H^{\beta,q}_w(\Omega) \hookrightarrow L^r_w(\Omega)$. If $q \le r$ this follows from Lemma 10.1.4 and if $q > r$ then one obtains from the definition of the spaces $H^{\beta,q}_w(\Omega) \hookrightarrow L^q_w(\Omega) \hookrightarrow L^r_w(\Omega)$.
- $\frac{1}{r} + \frac{1}{r} + \frac{1}{t'} = 1$.
- $-\frac{1}{(t-1)t'} + \frac{1}{r} + \frac{1}{r} = 0$.

$$\begin{aligned}\left|\int uv\phi\,dx\right| &= \left|\int uw^{\frac{1}{r}}vw^{\frac{1}{r}}\psi w_t^{\frac{1}{t'}}\,dx\right| \\ &\le \|u\|_{r,w}\|v\|_{r,w}\|\psi\|_{t',w_t} \le c\|u\|_{\beta,q,w}\|v\|_{\beta,q,w}\|\psi\|_{t',w_t}.\end{aligned}$$

1. b) First we assume that $\beta < \frac{ns}{q}$. We set $r = \frac{nsq}{-q\beta+ns}$ and $\eta = \left(1-\frac{1}{r}-\frac{1}{t}\right)^{-1} = \frac{rt}{rt-t-r}$. Then

- $\eta' = \frac{rt}{r+t} = \frac{nsq}{q+2ns-2q\beta} > \frac{nqs}{3q} \ge s$ if $n \ge 3$. If $n = 2$ one needs the stronger assumption on β to ensure $\eta' \ge s$.
- $-\frac{1}{\eta'} + \frac{1}{t} + \frac{1}{ns} = -\frac{1}{r} + \frac{1}{ns} = \frac{1+\beta-\frac{ns}{q}}{ns} > 0$. Hence $H^{1,t'}_{w_t}(\Omega) \hookrightarrow L^\eta_{w_{\eta'}}(\Omega)$.
- $\frac{1}{t} + \frac{1}{r} + \frac{1}{\eta} = 1$ and $-\frac{1}{(\eta'-1)\eta} + \frac{1}{t} + \frac{1}{r} = 0$.

Thus we can estimate

$$\begin{aligned}\left|\int ku\phi\,dx\right| &= \left|\int kw^{\frac{1}{t}}uw^{\frac{1}{r}}\phi w_{\eta'}^{\frac{1}{\eta}}\,dx\right| \\ &\le \|k\|_{t,w}\|u\|_{r,w}\|\phi\|_{\eta,w_{\eta'}} \le c\|k\|_{t,w}\|u\|_{\beta,q,w}\|\phi\|_{1,t',w_t}.\end{aligned}$$

If $\beta \ge \frac{ns}{q}$ then $H^{\beta,q}_w(\Omega) \hookrightarrow L^r_w(\Omega)$ for every $r \in (1,\infty)$. Moreover, we find some $\eta > t'$ such that $H^{1,t'}_{w_t}(\Omega) \hookrightarrow L^\eta_{w_{\eta'}}(\Omega)$. Choosing r such that $\frac{1}{r} + \frac{1}{\eta} + \frac{1}{t} = 1$ we can repeat the above estimate.
2. As above we begin with the case $\beta < \frac{ns}{q}$. Let $\eta := \frac{nsq}{ns-q\beta}$, $\mu := \frac{nsq}{ns-q\beta+q}$ and $r := \frac{nsq}{2ns-2\beta q+q}$. Then one has

- $\frac{1}{r} = \frac{1}{\eta} + \frac{1}{\mu}$.
- $r > \frac{ns}{3} \geq s$ if $n \geq 3$. If $n = 2$ we need the stronger assumption on β to ensure $r > s$. Moreover, $\frac{1}{q} > \frac{1}{r} - \frac{2-\beta}{ns}$, thus $L^r_w(\Omega) \hookrightarrow H^{\beta-2,q}_w(\Omega)$.
- $\frac{1}{\eta} = \frac{1}{q} - \frac{\beta}{ns}$ which implies $H^{\beta,q}_w(\Omega) \hookrightarrow L^\eta_w(\Omega)$.
- $\frac{1}{q} - \frac{\beta-1}{ns} = \frac{1}{\mu}$ which shows $H^{\beta-1,q}_w(\Omega) \hookrightarrow L^\mu_w(\Omega)$.

Thus it follows

$$\begin{aligned} \|u\nabla v\|_{\beta-2,q,w} &\leq c\|u\nabla v\|_{r,w} = c\left(\int |u|^r w^{\frac{r}{\eta}} |\nabla v|^r w^{1-\frac{r}{\eta}} dx\right)^{\frac{1}{r}} \\ &\leq c\|u\|_{\eta,w}\|\nabla v\|_{\mu,w} \leq c\|u\|_{\beta,q,w}\|\nabla v\|_{\beta-1,q,w} \leq c\|u\|_{\beta,q,w}\|v\|_{\beta,q,w}. \end{aligned}$$

If $2 \geq \beta \geq \frac{ns}{q}$ then $H^{\beta,q}_w(\Omega) \hookrightarrow L^\eta_w(\Omega)$ for every $\eta \in (1,\infty)$. Thus if $\beta \neq 2$ we repeat the above estimate with r as above, $\mu = q$ and η such that $\frac{1}{\eta} + \frac{1}{\mu} = \frac{1}{r}$.

If $\beta = 2$ let $r = q$ and we may choose $\mu > q$ such that such that $H^{\beta-1,q}_w(\Omega) \hookrightarrow L^\mu_w(\Omega)$ and η such that $\frac{1}{\eta} + \frac{1}{\mu} = \frac{1}{r}$. $\square$

10.2 Stationary Navier-Stokes Equations in Bessel Potential Spaces

In this section we always assume

- $\Omega \subset \mathbb{R}^n$ is a bounded $C^{1,1}$-domain,
- $1 < q < \infty$ and $w \in A_s$ for some $1 \leq s < q$,
- $\beta \in [0,2]$ with $\frac{ns}{q} - 1 < \beta$.

If $n \leq 3$ one can always choose such a β since by Lemma 3.1.2 for every $w \in A_q$ there exists s as above with $s < q$ and $w \in A_s$. Thus $\frac{ns}{q} - 1 < n - 1 \leq 2$.

Definition 10.2.1. Let $0 \leq \beta \leq 2$, $1 < q < \infty$ and $w \in A_q$. Moreover, let $g \in T^{\beta,q}_w(\partial\Omega)$, $F \in Y^{\beta-2,q}_w(\Omega)$ and $K \in L^t_w(\Omega)$. Then $u \in H^{\beta,q}_w(\Omega)$ is called a very weak solution to the stationary Navier-Stokes equations, if

$$-\langle u, \Delta\phi\rangle - \langle uu, \nabla\phi\rangle - \langle Ku, \phi\rangle = \langle F, \phi\rangle - \langle g, N\cdot\nabla\phi\rangle_{\partial\Omega} \quad \text{for every} \quad \phi \in Y^{2,q'}_{w',\sigma}(\Omega),$$

$\operatorname{div} u = K$ is fulfilled in the sense of distributions and $u \cdot N|_{\partial\Omega} = g \cdot N$ in the sense of (6.3.4).

Theorem 10.2.2. *Let $q > 1$, $w \in A_s$ for some $1 \leq s < q$, $0 \leq \beta < 1$ and $\beta > \frac{ns}{q} - 1$ if $n \geq 3$ and $\beta > -\frac{1}{2} + \frac{2s}{q}$ if $n = 2$. Moreover, let $F \in Y^{\beta-2,q}_w(\Omega)$, $K \in L^t_w(\Omega)$ with*

$$\frac{1-\beta}{ns} + \frac{1}{q} - \frac{1}{t} = 0 \tag{10.2.1}$$

and $g \in T_w^{\beta,q}(\partial\Omega)$ with $\langle K, 1\rangle_\Omega = \langle g, N\rangle_{\partial\Omega}$. Then there exists a constant $\rho > 0$ independent of the data such that, if

$$\|F\|_{Y_w^{\beta-2,q}} + \|K\|_{t,w} + \|g\|_{T_w^{\beta,q}(\partial\Omega)} \le \rho,$$

then there exists a very weak solution $u \in H_w^{\beta,q}(\Omega)$ to the stationary Navier-Stokes equations. This solution satisfies the estimate

$$\|u\|_{\beta,q,w} \le c\left(\|F\|_{-1,t,w} + \|K\|_{t,w} + \|g\|_{T_w^{\beta,q}(\partial\Omega)}\right) \tag{10.2.2}$$

with $c = c(\beta, q, w, \Omega) > 0$. Furthermore, if we assume in addition that $F \in W_w^{-1,t}(\Omega)$, then u fulfills $u|_{\partial\Omega} = g$ in the sense of (6.3.6).

Proof. By the Lemmas 10.1.4 and 10.1.5 one has

$$L_w^t(\Omega) \hookrightarrow H_{w,0}^{\beta-1,q}(\Omega) \quad\text{and}\quad W_w^{-1,t}(\Omega) \hookrightarrow Y_w^{\beta-2,q}(\Omega).$$

For $u \in H_w^{\beta,q}(\Omega)$ let $W(u) \in (C_0^\infty(\Omega))'$ be given by

$$\langle W(u), \phi\rangle = \langle uu, \nabla\phi\rangle + \langle Ku, \phi\rangle \quad\text{for all}\quad \phi \in C_0^\infty(\Omega).$$

By Lemma 10.1.5.1 one has for $\phi \in C_0^\infty(\Omega)$

$$\begin{aligned} |\langle W(u), \phi\rangle| &\le c\|u\|_{\beta,q,w}^2\|\nabla\phi\|_{t',w'} + c\|K\|_{t,w}\|u\|_{\beta,q,w}\|\phi\|_{1,t',w'} \\ &\le c(\|u\|_{\beta,q,w}^2 + \|K\|_{t,w}\|u\|_{\beta,q,w})\|\phi\|_{1,t',w'} \end{aligned}$$

and hence $W(u) \in W_w^{-1,t}(\Omega) \hookrightarrow Y_w^{\beta-2,q}(\Omega)$ with

$$\|W(u)\|_{Y_w^{\beta-2,q}} \le c_1\|W(u)\|_{-1,t,w} \le c(\|u\|_{\beta,q,w}^2 + \|K\|_{t,w}\|u\|_{\beta,q,w}). \tag{10.2.3}$$

We define the mapping $S : H_w^{\beta,q}(\Omega) \to H_w^{\beta,q}(\Omega)$ by

$$\begin{aligned} -\langle Su, \Delta\phi\rangle &= \langle F, \phi\rangle + \langle W(u), \phi\rangle - \langle g, N\cdot\nabla\phi\rangle_{\partial\Omega} && \text{for every}\quad \phi \in Y_{w',\sigma}^{2,q'}(\Omega), \\ -\langle Su, \nabla\psi\rangle &= \langle K, \psi\rangle - \langle g, N\psi\rangle_{\partial\Omega} && \text{for every}\quad \psi \in W_{w'}^{1,q'}(\Omega). \end{aligned}$$

The operator S is well-defined by Theorem 8.1.4.

We want to use Banach's Fixed Point Theorem to show that S has a fixed point under the assumption that the data is small enough.

By the a priori estimate in Theorem 8.1.4 we know that

$$\|v\|_{\beta,q,w} \le D(\|F\|_{Y_w^{\beta-2,q}} + \|K\|_{t,w} + \|g\|_{T_w^{\beta,q}(\partial\Omega)}), \tag{10.2.4}$$

if v is a very weak solution to the Stokes problem with respect to the data $F \in Y_w^{\beta-2,q}(\Omega)$, $K \in L_w^t(\Omega)$ and $g \in T_w^{\beta,q}(\partial\Omega)$.

We assume that the data F, K and g are chosen small enough such that the right hand side of (10.2.4) is strictly smaller then $\rho := \frac{1}{6cD}$, where c is the constant in the estimate (10.2.3) and D is the constant in the a priori estimate (10.2.4). Without loss of generality we assume that $D \ge 1$, which implies that additionally $\|K\|_{t,w} < \rho$.

Next we show that for such data and $\delta = \frac{2}{6cD}$ the ball closed ball $\overline{B_\delta(0)}$ in $H^{\beta,q}_w(\Omega)$ is mapped by S into itself. By (10.2.3) and (10.2.4) one has for $u \in B_\delta(0)$

$$\begin{aligned}
\|Su\|_{\beta,q,w} &\leq D\left(\|F\|_{Y^{\beta-2,q}_w} + c(\|u\|^2_{\beta,q,w} + \|K\|_{t,w}\|u\|_{\beta,q,w}) + \|K\|_{t,w} + \|g\|_{T^{\beta,q}_w(\partial\Omega)}\right) \\
&< \rho + cD\rho\delta + cD\delta^2 = \frac{6cD + 2cD + 4cD}{(6cD)^2} = \frac{2}{6cD} = \delta.
\end{aligned}$$

The next step is to show that S is a contraction on $B_\delta(0)$. Take $u, v \in B_\delta(0)$. Then $Su - Sv$ is a solution of

$$\begin{aligned}
-\langle Su - Sv, \Delta\phi\rangle &= \langle W(u) - W(v), \phi\rangle \quad &&\text{for every } \phi \in Y^{2,q'}_{w',\sigma}(\Omega) \\
-\langle Su - Sv, \nabla\psi\rangle &= 0 &&\text{for every } \psi \in W^{1,q'}_{w'}(\Omega).
\end{aligned}$$

Moreover, from Lemma 10.1.5.1 we obtain

$$\begin{aligned}
|\langle W(u) - W(v), \phi\rangle| &\leq |\langle (u-v)u, \nabla\phi\rangle| + |\langle v(u-v), \nabla\phi\rangle| + |\langle K(u-v), \phi\rangle| \\
&\leq c(\|u\|_{\beta,q,w} + \|v\|_{\beta,q,w} + \|K\|_{t,w})\|u-v\|_{\beta,q,w}\|\phi\|_{1,t',w_t} \\
&\leq (2c\delta + c\rho)\|u-v\|_{\beta,q,w}\|\phi\|_{1,t',w_t} \\
&= \frac{5}{6D}\|u-v\|_{\beta,q,w}\|\phi\|_{1,t',w_t}.
\end{aligned}$$

Thus we obtain from the a priori estimate (10.2.4) that

$$\|Su - Sv\|_{\beta,q,w} \leq D\|W(u) - W(v)\|_{-1,t,w} \leq \frac{5}{6}\|u-v\|_{\beta,q,w}.$$

Now Banach's fixed point theorem gives us the existence of a unique fixed point of S within the ball $B_\delta(0)$ and hence of a solution $u \in H^{\beta,q}_w(\Omega)$ to the stationary Navier-Stokes system.

The a priori estimate (10.2.2) follows from

$$\begin{aligned}
\|u\|_{\beta,q,w} &= \|S(u)\|_{\beta,q,w} \\
&\leq D\left(\|F\|_{Y^{\beta-2,q}_w} + \|K\|_{t,w} + \|g\|_{T^{\beta,q}_w(\partial\Omega)} + c(\|u\|^2_{\beta,q,w} + \|K\|_{t,w}\|u\|_{\beta,q,w})\right)
\end{aligned}$$

since $Dc(\|u\|_{\beta,q,w} + \|K\|_{t,w}) \leq \frac{3}{6}$ and we may subtract $\frac{3}{6}\|u\|_{\beta,q,w}$ from both sides of the above equation.

Now assume that $F \in W^{-1,q}_w(\Omega)$. It remains to show that in this case the solution u fulfills the boundary condition $u|_{\partial\Omega} = g$. To see this one uses the fact that u is a very weak solution to the Stokes equations with respect to the data

$$\begin{aligned}
f &= [\phi \;\mapsto \langle F, \phi\rangle + \langle W(u), \phi\rangle - \langle g, N\cdot\nabla\phi\rangle_{\partial\Omega}] \\
k &= [\psi \;\mapsto \langle K, \psi\rangle - \langle g, N\psi\rangle_{\partial\Omega}],
\end{aligned}$$

where $f|_{C^\infty_0(\Omega)} = [\phi \mapsto \langle F, \phi\rangle + \langle W(u), \phi\rangle] \in W^{-1,t}_w(\Omega)$. Then the assertion about the boundary values follows from Theorem 8.1.4. $\square$

Definition 10.2.3. Let $1 \le \beta \le 2$. Moreover, let $F \in H_w^{\beta-2,q}(\Omega)$, $K \in H_w^{\beta-1,q}(\Omega)$ and $g \in T_w^{\beta,q}(\partial\Omega)$. Then $u \in H_w^{\beta,q}(\Omega)$ is called a weak solution to the stationary Navier-Stokes equations, if

$$\langle \nabla u, \nabla \phi \rangle + \langle u \cdot \nabla u, \phi \rangle = \langle F, \phi \rangle \quad \text{for every} \quad \phi \in C_{0,\sigma}^\infty(\Omega),$$

$\operatorname{div} u = K$ and $u|_{\partial\Omega} = g$.

Theorem 10.2.4. *Let $1 \le \beta \le 2$ and $\beta > \frac{ns}{q} - 1$ if $n \ge 3$ and $\beta > \frac{2s}{q} - \frac{1}{2}$ if $n = 2$. Moreover, let $F \in H_w^{\beta-2,q}(\Omega)$, $K \in H_w^{\beta-1,q}(\Omega)$ and $g \in T_w^{\beta,q}(\partial\Omega)$ with $\int K\,dx = \int_{\partial\Omega} gN\,dS$. Then there exists a constant $\rho > 0$ such that, if*

$$\|F\|_{\beta-2,q,w} + \|K\|_{\beta-1,q,w} + \|g\|_{T_w^{\beta,q}(\partial\Omega)} \le \rho,$$

then there exists a weak solution $u \in H_w^{\beta,q}(\Omega)$ to the stationary Navier-Stokes equations. This solution satisfies the estimate

$$\|u\|_{\beta,q,w} \le c(\|F\|_{\beta-2,q,w} + \|K\|_{\beta-1,q,w} + \|g\|_{T_w^{\beta,q}(\partial\Omega)})$$

with $c = c(\beta, q, w, \Omega) > 0$.

Proof. This can be proved in the same way as Theorem 10.2.2 using Lemma 10.1.5.2. instead of Lemma 10.1.5.1. and Theorem 8.1.3 instead of Theorem 8.1.4. □

The very weak solution is unique even without the assumption of the smallness of the exterior force f and the boundary condition g. This follows from the uniqueness of very weak solutions to the stationary Navier-Stokes equations in the unweighted case which has been proved in [16] in the case $n \ge 3$. This is shown in the following theorem.

Theorem 10.2.5. *Let the data F, K and g be given as in Theorem 10.2.2, Theorem 10.2.4, respectively, and let u be a very weak solution to the stationary Navier-Stokes system with respect to the data F, K and g.*

Then there exists a constant $\rho > 0$ such that under the condition that

$$\|u\|_{\beta,q,w} + \|K\|_{t,w} \le \rho$$

there exists at most one very weak solution to the stationary Navier-Stokes equations according to Definition 10.2.1.

Proof. By Lemma 10.1.4 and Lemma 3.2.2 one has for $\beta < \frac{ns}{q}$

$$u \in H_w^{\beta,q}(\Omega) \hookrightarrow L_w^{\frac{nsq}{-q\beta+ns}}(\Omega) \hookrightarrow L^{\frac{nq}{-q\beta+ns}}(\Omega) = L^\eta(\Omega),$$

where, by the assumptions on β, one has $\eta := \frac{nq}{-q\beta+ns} > n$.

For $\beta \ge \frac{ns}{q}$ the embedding $H_w^{\beta,q}(\Omega) \hookrightarrow L_w^\mu(\Omega)$ holds for every $\mu > 1$. If we choose $\mu = \eta s$ with $\eta > n$, then we obtain that also in this case

$$H_w^{\beta,q}(\Omega) \hookrightarrow L^\eta(\Omega) \tag{10.2.5}$$

We want to show that $\eta > n$ in (10.2.5) can be chosen such that

$$K \in L^{\frac{\eta n}{\eta+n}}(\Omega) \quad \text{and} \quad F \in W^{-1,\frac{n\eta}{\eta+n}}(\Omega)$$

is fulfilled additionally. If $\beta \leq 1$ then one has by assumption

$$K \in L^t_w(\Omega) \quad \text{and} \quad F \in W^{-1,t}_w(\Omega)$$

and by the proof of Lemma 10.1.5 one has $t > \frac{ns}{2} = \frac{n^2 s}{n+n}$. Thus we find η with the asserted properties, since again by Lemma 3.2.2 one has the embeddings

$$L^t_w(\Omega) \hookrightarrow L^{\frac{t}{s}}(\Omega) \quad \text{and} \quad W^{-1,t}_w(\Omega) \hookrightarrow F \in W^{-1,\frac{t}{s}}(\Omega).$$

Now let $\beta > 1$. Then the embedding $H^{\beta-1,q}_w(\Omega) \hookrightarrow L^t_w(\Omega)$ follows directly from Lemma 10.1.4 and $Y^{\beta-2,q}_w(\Omega) \hookrightarrow W^{-1,t}_w(\Omega)$ follows when taking the dual spaces in the embedding $W^{1,t'}_{w',0}(\Omega) \hookrightarrow Y^{2-\beta,q'}_{w'}(\Omega)$, that is shown in Lemma 10.1.4.

Moreover, from Corollary 8.1.6 we obtain that $g \in W^{-\frac{1}{\eta},\eta}(\partial\Omega) := T^{0,\eta}_1(\partial\Omega)$. Hence data and solution are contained in the same spaces as in [16, Theorem 1.5]. Thus exactly the same proof as given there can be used to show that two solutions that correspond to the same data coincide. □

11 Instationary Navier-Stokes Equations in Weighted Bessel Potential spaces

11.1 Definition and Discussions

We consider the Navier-Stokes equations with inhomogeneous data

$$\begin{aligned} \partial_t u - \Delta u + u\nabla u + \nabla p &= f && \text{in } (0,T)\times\Omega \\ \operatorname{div} u &= k && \text{in } (0,T)\times\Omega \\ u &= g && \text{on } (0,T)\times\partial\Omega \\ u(0) &= u_0 && \text{in } \Omega \end{aligned}$$

on a bounded $C^{1,1}$-domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, and with $T \in (0,\infty]$. As before multiplication with a test function and formal integration by parts lead to the definition of very weak solutions below.

As seen in the stationary case, the estimates of the nonlinear term in the weighted context require higher regularity in space. In particular, the divergence term k has to be given by a function to assure that the multiplication $u\cdot\nabla u$ is well-defined. Moreover, since Sobolev-like inequalities require strong assumptions on the weight function, we construct the solution in weighted Bessel potential spaces. However, when treating the instationary linear case, we have seen that this requires some stronger time regularity of the divergence and the boundary conditions.

This is the reason for the choice of the spaces for the data in the definition below.

Definition 11.1.1. Let $\beta \in [0,2]$, $r,q \in (1,\infty)$, $w \in A_q$. Moreover, in the case $\beta \leq 1$ choose $\mu > 1$ such that

$$L^\mu_w(\Omega) \hookrightarrow H^{\beta-1,q}_{w,0}(\Omega).$$

Take

$$\begin{aligned} &f \in L^r(0,T;Y^{\beta-2,q}_w(\Omega)),\\ &k \in L^r(0,T;L^\mu_w(\Omega)) \cap H_0^{\frac{\beta}{2},r}((0,T];W^{-1,q}_{w,0}(\Omega)) \quad \text{if} \;\; \beta < 1 \;\; \text{and}\\ &k \in L^r(0,T;H^{\beta-1,q}_w(\Omega)) \cap H_0^{\frac{\beta}{2},r}((0,T];W^{-1,q}_{w,0}(\Omega)) \quad \text{if} \;\; \beta \geq 1,\\ &g \in L^r(0,T;T^{\beta,q}_w(\partial\Omega)) \cap H_0^{\frac{\beta}{2},r}((0,T];T^{0,q}_w(\partial\Omega)),\\ &u_0 \in \mathcal{I}^{\beta,q,r}_w(\Omega). \end{aligned}$$

Then $u \in L^r(0,T; H_w^{\beta,q}(\Omega))$ is called a very weak solution to the Navier-Stokes problem if

$$\begin{aligned} &-\langle u, \phi_t\rangle_{\Omega,T} - \langle u, \Delta\phi\rangle_{\Omega,T} \\ &\qquad\qquad = \langle f, \phi\rangle_{\Omega,T} - \langle g, N\cdot\nabla\phi\rangle_{\partial\Omega,T} + \langle uu, \nabla\phi\rangle_{\Omega,T} + \langle ku, \phi\rangle_{\Omega,T} - \langle u_0, \phi(0)\rangle_\Omega \end{aligned}$$

for every $\phi \in W^{1,r'}(0,T; Y^{2,q'}_{w',\sigma}(\Omega))$ with $\operatorname{supp}\phi \subset [0,T) \times \overline{\Omega}$, $\operatorname{div} u = k$ is fulfilled in the sense of distributions and $u\cdot N|_{\partial\Omega} = g\cdot N$ in the sense of (6.3.4).

11.2 Existence and Uniqueness

In the unweighted case the boundedness of imaginary powers of the Stokes operator is used to prove an exact characterization of the domains of fractional powers of the Stokes operator, see Giga [33]. However, in weighted function spaces this is not established. We use the following Theorem by Franzke [23] as a replacement.

Theorem 11.2.1. *Let X be a Banach space and A a densely defined positive operator in X, i.e.,*

$$\|(\lambda + A)^{-1}\| \le \frac{K}{1+\lambda} \quad \textit{for every} \quad \lambda \ge 0.$$

Then for $m \in \mathbb{N}$, $0 < \theta < 1$ and $0 < \alpha_- < \theta m < \alpha_+$ one has

$$\mathcal{D}(A^{\alpha_+}) \hookrightarrow [X, \mathcal{D}(A^m)]_\theta \hookrightarrow \mathcal{D}(A^{\alpha_-}).$$

In particular if $A = \mathcal{A}$ is the Stokes operator in $L^q_{w,\sigma}(\Omega)$, $m = 1$, $0 < \beta < 2$ and $\varepsilon > 0$, then we find by Corollary 8.2.2

$$\mathcal{D}(\mathcal{A}^{\frac{1}{2}\beta+\varepsilon}) \hookrightarrow Y^{\beta,q}_{w,\sigma}(\Omega) \hookrightarrow \mathcal{D}(\mathcal{A}^{\frac{1}{2}\beta-\varepsilon}). \tag{11.2.1}$$

Thus one has the estimate

$$c_1\|A^{\frac{1}{2}\beta-\varepsilon}u\|_{q,w} \le \|u\|_{Y^{\beta,q}_{w,\sigma}} \le c_2\|A^{\frac{1}{2}\beta+\varepsilon}u\|_{q,w}.$$

Moreover, if we consider the generalized Stokes operator in $Y^{-1,\rho}_{w,\sigma}(\Omega)$, one obtains by Corollary 8.2.2

$$\begin{aligned} \|u\|_{\rho,w} &\le c\|\mathcal{A}u\|_{Y^{-2,\rho}_{w,\sigma}} = c\|\mathcal{A}^{\frac{1}{2}-\varepsilon}\mathcal{A}^{\frac{1}{2}+\varepsilon}u\|_{Y^{-2,\rho}_{w,\sigma}} \\ &\le c\|\mathcal{A}^{\frac{1}{2}+\varepsilon}u\|_{[Y^{-2,\rho}_{w,\sigma}, L^q_{w,\sigma}]_{\frac{1}{2}}} \le c\|\mathcal{A}^{\frac{1}{2}+\varepsilon}u\|_{Y^{-1,\rho}_{w,\sigma}} \\ &= c\|\mathcal{A}^{\frac{1}{2}+\varepsilon}u\|_{H^{-1,\rho}_{w,\sigma}}. \end{aligned} \tag{11.2.2}$$

The proof of existence and uniqueness of very weak solutions to the instationary Navier-Stokes equations requires the Variations of Constants Formula established in the following lemma.

Lemma 11.2.2. *Let $1 < q, r < \infty$, $0 \le \beta \le 2$. Moreover, take $f \in L^r(0,T; Y^{\beta-2,q}_{w,\sigma}(\Omega))$ and let $u \in L^r(0,T; Y^{\beta,q}_{w,\sigma}(\Omega))$ be the solution to*

$$u_t + \mathcal{A}u = f \quad \textit{in} \quad \mathcal{D}'(0,T; Y^{\beta-2,q}_{w,\sigma}(\Omega)) \quad \textit{and} \quad u(0) = 0,$$

where $\mathcal{A} = \mathcal{A}_{\beta-2,q,w}$ *is the generalized Stokes operator in* $Y^{\beta-2,q}_{w,\sigma}(\Omega)$. *Then*

$$u(t) = \int_0^t e^{-\mathcal{A}(t-\tau)} f(\tau)\, d\tau \quad \textit{for almost every } t \in (0,T).$$

Proof. As in Lemma 8.2.6 we write $\mathcal{A}_{\beta-2,q,w}$ for the Stokes operator in $Y^{\beta-2,q}_{w,\sigma}(\Omega)$. From the embeddings $H^{\beta,q}_{w,\sigma}(\Omega) \hookrightarrow L^q_{w,\sigma}(\Omega)$ and $Y^{\beta-2,q}_{w,\sigma}(\Omega) \hookrightarrow Y^{-2,q}_{w,\sigma}(\Omega)$ we know that

$$u \in L^r(0,T; L^q_{w,\sigma}(\Omega)) \quad \text{and} \quad f \in L^r(0,T; Y^{-2,q}_{w,\sigma}(\Omega)).$$

Thus we obtain

$$\mathcal{A}^{-1}_{0,q,w} u \in L^r(0,T; Y^{2,q}_{w,\sigma}(\Omega)) \quad \text{and} \quad \mathcal{A}^{-1}_{-2,q,w} f \in L^r(0,T; L^q_{w,\sigma}(\Omega)).$$

Then using Lemma 8.2.6 we obtain that $\mathcal{A}^{-1}_{-2,q,w} u = \mathcal{A}^{-1}_{0,q,w} u$ is the strong solution to the instationary Stokes problem

$$(\mathcal{A}^{-1}_{0,q,w} u)_t + \mathcal{A}_{0,q,w}(\mathcal{A}^{-1}_{0,q,w} u) = \mathcal{A}^{-1}_{-2,q,w} f.$$

By the Variation of Constants formula in the case of strong solutions [25] one obtains

$$\mathcal{A}^{-1}_{0,q,w} u(t) = \int_0^t e^{-\mathcal{A}_{0,q,w}(t-\tau)} \mathcal{A}^{-1}_{-2,q,w} f(\tau)\, d\tau. \tag{11.2.3}$$

Moreover, by Lemma 8.2.6 one has

$$\begin{aligned} \mathcal{A}_{0,q,w} e^{-(t-\tau)\mathcal{A}_{0,q,w}} \mathcal{A}^{-1}_{-2,q,w} f &= \mathcal{A}_{-2,q,w} e^{-(t-\tau)\mathcal{A}_{-2,q,w}} \mathcal{A}^{-1}_{-2,q,w} f = e^{-(t-\tau)\mathcal{A}_{-2,q,w}} f \\ &= e^{-(t-\tau)\mathcal{A}_{\beta-2,q,w}} f. \end{aligned}$$

Thus if one applies $\mathcal{A}_{0,q,w}$ to both sides of (11.2.3) the proof of the lemma is finished. □

Theorem 11.2.3. *Let* $\beta \in [0,2]$ *with* $\beta > \frac{ns}{q} - 1$, *where* $q \in (1,\infty)$ *and* $w \in A_s$ *for some* $s < q$. *Moreover, let* $\Omega \subset \mathbb{R}^n$, $n \geq 2$ *be a bounded* $C^{1,1}$*-domain, if* $\beta \leq 1$, *and a bounded* $C^{2,1}$*-domain, if* $\beta > 1$.

Choose $r \in (1,\infty)$ *such that*

$$\begin{aligned} &\frac{1}{r} < \min\left\{ -\frac{ns}{2q} + \frac{\beta}{2} + \frac{1}{2}, \frac{1-\beta}{2} \right\} \quad \textit{if } 0 \leq \beta < 1, \\ &\frac{1}{r} < \min\left\{ -\frac{ns}{2q} + \frac{\beta}{2} + \frac{1}{2}, \frac{2-\beta}{2} \right\} \quad \textit{if } 1 \leq \beta < 2 \textit{ and} \\ &\frac{1}{r} < \min\left\{ -\frac{ns}{2q} + \frac{3}{2}, \frac{1}{2} \right\} \quad \textit{and} \quad \frac{2}{q} - \frac{2}{ns} < \frac{1}{s} \quad \textit{if } \beta = 2. \end{aligned}$$

In the case $n = 2$ *and* $\beta \in [1,2)$ *we assume in addition that* $\beta > \frac{2s}{q} - \frac{1}{2}$.

Take f, k, g *and* u_0 *as in Definition 11.1.1 with* μ *chosen such that*

$$\frac{1-\beta}{ns} + \frac{1}{q} - \frac{1}{\mu} = 0 \quad \textit{in the case} \quad \beta \leq 1.$$

Then, if $\beta \le 1$ there exists a constant $\eta = \eta(\Omega, \beta, q, w, r) > 0$ with the following property: If $0 < T' \le T$ with

$$\left(\int_0^{T'} \|e^{-\tau\mathcal{A}}u_0\|_{\beta,q,w}^r d\tau\right)^{\frac{1}{r}} + \|f\|_{L^r(0,T';Y_w^{\beta-2,q})}$$
$$+\|k\|_{L^r(0,T';L^\mu_w)\cap H_0^{\frac{\beta}{2},r}((0,T'];W^{-1,q}_{w,0})} + \|g\|_{L^r(0,T';T_w^{\beta,q})\cap H_0^{\frac{\beta}{2},r}((0,T'];T_w^{0,q})} \le \eta,$$

then there exists a unique very weak solution $u \in L^r(0,T'; H_w^{\beta,q}(\Omega))$ to the Navier-Stokes equations. For every $T'' \in (0,T']$, $T'' < \infty$ this solution u satisfies the estimate

$$\begin{aligned}\|u\|_{L^r(0,T';H_w^{\beta,q})} + &\|u_t|_{Y^{2,q'}_{w',\sigma}(\Omega)}\|_{L^{\frac{r}{2}}(0,T'';Y^{\beta-2,q}_{w,\sigma}(\Omega))}\\ \le c\Big(&\Big(\int_0^{T'} \|e^{-\tau\mathcal{A}}u_0\|_{\beta,q,w}^r d\tau\Big)^{\frac{1}{r}} + \|f\|_{L^r(0,T';Y_w^{\beta-2,q})}\\ &+ \|k\|_{L^r(0,T';L^\mu_w)\cap H_0^{\frac{\beta}{2},r}((0,T'];W^{-1,q}_{w,0})} + \|g\|_{L^r(0,T';T_w^{\beta,q})\cap H_0^{\frac{\beta}{2},r}((0,T'];T_w^{0,q})}\Big);\end{aligned} \tag{11.2.4}$$

here c increases with increasing T'' but can be chosen independently of T and T'.

If $\beta > 1$ then the same assertion holds if $L^r(0,T'; L^\mu_w(\Omega)) \cap H_0^{\frac{\beta}{2},r}((0,T']; W^{-1,q}_{w,0}(\Omega))$ is replaced by $L^r(0,T'; H_w^{\beta-1,q}(\Omega)) \cap H_0^{\frac{\beta}{2},r}((0,T']; W^{-1,q}_{w,0}(\Omega))$.

Proof. Let $E \in L^r(0,T; H_w^{\beta,q}(\Omega))$ be the very weak solution to the instationary Stokes problem with respect to the data f, k, g and u_0 in the sense of Theorem 9.5.4.

Assume that $u \in L^r(0,T; H_w^{\beta,q}(\Omega))$ is the very weak solution to the Navier-Stokes equations we are looking for. Then $\tilde{u} := u - E$ solves

$$\partial_t \tilde{u} - \Delta\tilde{u} + \nabla\tilde{p} = -W(u), \quad \operatorname{div}\tilde{u} = 0, \quad \tilde{u}|_{\partial\Omega} = 0 \ \text{ and } \ \tilde{u}(0) = 0.$$

in the very weak sense with

$$W(u)(t) := [Y^{2,q'}_{w',\sigma}(\Omega) \ni \phi \mapsto -\langle u(t)u(t), \nabla\phi\rangle_\Omega - \langle k(t)u(t), \phi\rangle_\Omega]$$

for almost every t. This means

$$-\langle \tilde{u}, \phi_t\rangle_{\Omega,T} - \langle \tilde{u}, \Delta\phi\rangle_{\Omega,T} = \langle W(u), \phi\rangle_\Omega$$

with ϕ as in Definition 11.1.1. Then the Variation of Constants Formula proved in Lemma 11.2.2 yields

$$\tilde{u}(t) = -\int_0^t e^{-(t-\tau)\mathcal{A}} W(u) d\tau =: \mathcal{G}(\tilde{u})(t).$$

As a first step we assume $\beta < 1$. By the definition of μ and the assumptions on β one has $s \le \mu < q$ and by Lemma 10.1.4 one obtains $L^\mu_w(\Omega) \hookrightarrow H_w^{\beta-1,q}(\Omega)$. Put

$$\alpha = \frac{1}{r} - 1 < 0 \quad \text{and} \quad \varepsilon := \min\left\{\frac{1}{5}\left(-\alpha - \frac{\beta}{2} - \frac{1}{2}\right), \frac{1}{5}\left(-\frac{ns}{q} + \beta + 1 - \frac{2}{r}\right)\right\} > 0,$$

where ε is positive by the assumption on r. Moreover, if $\beta < \frac{ns}{q}$ we set $\rho := \frac{nsq}{2ns - 2q\beta}$. Then one obtains by the assumptions on β:

- ρ is well-defined and $\rho > \frac{ns}{2} \geq s$.
- $\frac{1}{q} \geq \frac{1}{\rho} - \frac{-2\alpha-5\varepsilon-\beta-1}{ns}$ with $0 < -2\alpha - 5\varepsilon - \beta - 1 \leq 1 < n$.
- $\frac{1}{2\rho} = \frac{1}{q} - \frac{\beta}{ns}$.

This proves

$$H_w^{\beta,q}(\Omega) \hookrightarrow L_w^{2\rho}(\Omega) \quad \text{and} \quad H_w^{-2\alpha-5\varepsilon-\beta-1,\rho}(\Omega) \hookrightarrow L_w^q(\Omega), \tag{11.2.5}$$

using Lemma 10.1.4 if $\rho < q$. For $q \leq \rho$ the latter embedding is obvious.

If $1 > \beta \geq \frac{ns}{q}$ then $H_w^{\beta,q}(\Omega) \hookrightarrow L_w^{2\rho}(\Omega)$ for every $\rho \in (1,\infty)$. In this case we choose ρ with $\frac{1}{q} < \frac{1}{\rho} < \frac{1}{q} + \frac{-2\alpha-5\varepsilon-\beta-1}{ns}$. Then the embeddings (11.2.5) hold for ρ.

By (11.2.1), (11.2.2) and Theorem 2.2.3 one obtains the estimate

$$\begin{aligned}
\|\mathcal{G}(\tilde{u})(t)\|_{H_w^{\beta,q}} \leq& \|\mathcal{G}(\tilde{u})(t)\|_{Y_w^{\beta,q}} \\
\leq& \int_0^t \left\|e^{-(t-\tau)\mathcal{A}} W(u)\right\|_{Y_w^{\beta,q}} d\tau \\
\leq& \int_0^t \left\|\mathcal{A}^{-\alpha} e^{-(t-\tau)\mathcal{A}} \mathcal{A}^{\alpha+\frac{\beta}{2}+\varepsilon} W(u)\right\|_{q,w} d\tau \\
\leq& c\int_0^t \frac{1}{(t-\tau)^{-\alpha}} \left\|\mathcal{A}^{(\alpha+2\varepsilon+\frac{\beta}{2}+\frac{1}{2})-\frac{1}{2}-\varepsilon} W(u)\right\|_{q,w} d\tau \\
\leq& c\int_0^t \frac{1}{(t-\tau)^{-\alpha}} \left\|\mathcal{A}^{(\alpha+2\varepsilon+\frac{\beta}{2}+\frac{1}{2})-\frac{1}{2}-\varepsilon} W(u)\right\|_{-2\alpha-5\varepsilon-\beta-1,\rho,w} d\tau \\
\leq& c\int_0^t \frac{1}{(t-\tau)^{-\alpha}} \left\|\mathcal{A}^{-\frac{1}{2}-\varepsilon} W(u)\right\|_{\rho,w} d\tau \\
\leq& c\int_0^t \frac{1}{(t-\tau)^{-\alpha}} \|W(u)\|_{-1,\rho,w} d\tau,
\end{aligned} \tag{11.2.6}$$

where we have used $\mathcal{A}_{-2,q,w}|_{Y_{w,\sigma}^{-1,\rho}} = \mathcal{A}_{-1,\rho,w}$, where $\mathcal{A}_{-2,q,w}$ is the generalized Stokes operator in $Y_{w,\sigma}^{-2,q}(\Omega)$ and $\mathcal{A}_{-1,\rho,w}$ is the generalized Stokes operator in $Y_{w,\sigma}^{-1,\rho}(\Omega) = (W_{0,w',\sigma}^{1,\rho'}(\Omega))'$.

We have to estimate $\|W(u)\|_{-1,\rho,w}$. For $\phi \in Y_{w',\sigma}^{2,q'}(\Omega)$ one has

$$\begin{aligned}
|\langle W(u),\phi\rangle_\Omega| \leq& |\langle uu, \nabla\phi\rangle_\Omega| + |\langle ku,\phi\rangle_\Omega| \\
\leq& \|uu\|_{\rho,w}\|\nabla\phi\|_{\rho',w_\rho} + \|ku\|_{\tilde{\rho},w}\|\phi\|_{\tilde{\rho}',w_{\tilde{\rho}}} \\
\leq& \left(\|u\|_{2\rho,w}^2 + \|k\|_{\mu,w}\|u\|_{2\rho,w}\right)\|\phi\|_{1,\rho',w_\rho} \\
\leq& c\left(\|u\|_{\beta,q,w}^2 + \|k\|_{\mu,w}\|u\|_{\beta,q,w}\right)\|\phi\|_{1,\rho',w_\rho}
\end{aligned} \tag{11.2.7}$$

since for $\tilde{\rho}$ given by $\frac{1}{\tilde{\rho}} = \frac{1}{\mu} + \frac{1}{2\rho}$ an elementary computation shows $\frac{1}{\rho} = \frac{1}{\tilde{\rho}} - \frac{1}{ns}$ which implies $H_{w_\rho}^{1,\rho'}(\Omega) \hookrightarrow L_{w_{\tilde{\rho}}}^{\tilde{\rho}'}(\Omega)$.

Thus we continue combining (11.2.6) and (11.2.7), the Hardy-Littlewood inequality [49, VIII 4.2] and the equality $\frac{1}{r}+\alpha+1=\frac{1}{\frac{r}{2}}$ Then we may estimate

$$\begin{aligned}
&\|\mathcal{G}(\tilde u)\|_{L^r(0,T;H^{\beta,q}_w(\Omega))}\\
&\le c\left\|\int_0^t \frac{1}{(t-\tau)^{-\alpha}}\left(\|u(\tau)\|^2_{\beta,q,w}+\|k(\tau)\|_{\mu,w}\|u(\tau)\|_{\beta,q,w}\right)d\tau\right\|_{L^r(0,T)}\\
&\le c\left\|\|u(\tau)\|^2_{\beta,q,w}+\|k(\tau)\|_{\mu,w}\|u(\tau)\|_{\beta,q,w}\right\|_{L^{\frac{r}{2}}(0,T)}\\
&\le c\left(\|u\|^2_{L^r(0,T;H^{\beta,q}_w(\Omega))}+\|k\|_{L^r(0,T;L^\mu_w(\Omega))}\|u\|_{L^r(0,T;H^{\beta,q}_w(\Omega))}\right)\\
&\le c\Big(\left(\|\tilde u\|_{L^r(0,T;H^{\beta,q}_w(\Omega))}+\|E\|_{L^r(0,T;H^{\beta,q}_w(\Omega))}\right)^2\\
&\quad+\|k\|_{L^r(0,T;L^\mu_w(\Omega))}\left(\|\tilde u\|_{L^r(0,T;H^{\beta,q}_w(\Omega))}+\|E\|_{L^r(0,T;H^{\beta,q}_w(\Omega))}\right)\Big).
\end{aligned} \tag{11.2.8}$$

Now assume

$$\begin{aligned}
&\left(\int_0^{T'}\|e^{-\tau\mathcal{A}}u_0\|^r_{\beta,q,w}d\tau\right)^{\frac{1}{r}}+\|f\|_{L^r(0,T';Y^{\beta-2,q}_w)}\\
&+\|k\|_{L^r(0,T';L^\mu_w)\cap H_0^{\frac{\beta}{2},r}((0,T'];W^{-1,q}_{w,0})}+\|g\|_{L^r(0,T';T^{\beta,q}_w)\cap H_0^{\frac{\beta}{2},r}((0,T'];T^{0,q}_w)}\le\eta
\end{aligned}$$

and $\|\tilde u\|_{L^r(0,T;H^{\beta,q}_w(\Omega))}<\delta$, where η and δ are positive but will be chosen sufficiently small later on. Then one has $\|E\|_{L^r(0,T;H^{\beta,q}_w(\Omega))}\le K\eta$, where K is the constant from the a priori estimate for the solution to the instationary Stokes equations, Theorem 9.5.4. Thus we obtain from (11.2.8)

$$\|\mathcal{G}(\tilde u)\|_{L^r(0,T;H^{\beta,q}_w(\Omega))}\le c((\delta+K\eta)^2+\eta(\delta+K\eta))<\delta,$$

if η and δ are sufficiently small. This shows $\mathcal{G}(B_\delta(0))\subset\overline{B_\delta(0)}$, where $\overline{B_\delta(0)}$ is the closed ball with radius δ in $L^r(0,T;H^{\beta,q}_w(\Omega))$.

We show that $\mathcal{G}$ is a contraction on $B_\delta(0)$. As above we estimate

$$\begin{aligned}
&|\langle W(E+\tilde u)-W(E+\tilde v),\phi\rangle_\Omega|\\
\le&|\langle (E+\tilde u)^2-(E+\tilde v)^2,\nabla\phi\rangle_\Omega|+|\langle k(E+\tilde u)-k(E+\tilde v),\phi\rangle|\\
\le&|\langle 2E(\tilde u-\tilde v),\nabla\phi\rangle_\Omega|+|\langle \tilde u\tilde u-\tilde v\tilde v,\nabla\phi\rangle_\Omega|+|\langle k(\tilde u-\tilde v),\phi\rangle_\Omega|\\
\le&c\big(\|E\|_{\beta,q,w}\|\tilde u-\tilde v\|_{\beta,q,w}+\|\tilde u\|_{\beta,q,w}\|\tilde u-\tilde v\|_{\beta,q,w}\\
&+\|\tilde v\|_{\beta,q,w}\|\tilde u-\tilde v\|_{\beta,q,w}+\|k\|_{\mu,w}\|\tilde u-\tilde v\|_{\beta,q,w}\big)\|\phi\|_{1,\rho',w_\rho}.
\end{aligned}$$

Thus

$$\|W(E+\tilde u)-W(E+\tilde v)\|_{-1,\rho,w}\le c\big(\|E\|_{\beta,q,w}+\|\tilde u\|_{\beta,q,w}+\|\tilde v\|_{\beta,q,w}+\|k\|_{\mu,w}\big)\|\tilde u-\tilde v\|_{\beta,q,w}.$$

Now an analogous estimate as for $\|\mathcal{G}(\tilde{u})\|_{L^r(0,T;H_w^{\beta,q}(\Omega))}$ shows

$$\begin{aligned}
&\|\mathcal{G}(\tilde{u})-\mathcal{G}(\tilde{v})\|_{L^r(0,T;H_w^{\beta,q}(\Omega))}\\
&\quad\le c\Big\|\int_0^t \frac{1}{(t-\tau)^{-\alpha}}\big(\|E\|_{\beta,q,w}+\|\tilde{u}\|_{\beta,q,w}+\|\tilde{v}\|_{\beta,q,w}+\|k\|_{\mu,w}\big)\|\tilde{u}-\tilde{v}\|_{\beta,q,w}d\tau\Big\|_{L^r(0,T')}\\
&\quad\le c\big(\|E\|_{L^r(0,T';H_w^{\beta,q})}+\|\tilde{u}\|_{L^r(0,T';H_w^{\beta,q})}+\|\tilde{v}\|_{L^r(0,T';H_w^{\beta,q})}+\|k\|_{L^r(0,T';L_w^{\mu})}\big)\\
&\quad\quad\cdot\|\tilde{u}-\tilde{v}\|_{L^r(0,T';H_w^{\beta,q})}\\
&\quad\le c(K\eta+\eta+2\delta)\|\tilde{u}-\tilde{v}\|_{L^r(0,T';H_w^{\beta,q})}.
\end{aligned}$$

This means, if η and δ are sufficiently small, then $\mathcal{G}$ is a contraction. Hence by Banach's fixed point theorem there exists a unique $\tilde{u}\in B_\delta(0)$ with $\mathcal{G}(\tilde{u})=\tilde{u}$. Then $u:=E+\tilde{u}$ is the solution we have been looking for.

We turn to the case $\beta\ge 1$. In this case the proof of existence follows the same lines of the case $\beta<1$ but using different embeddings. Moreover, the fact that $k\in L^r(0,T;H_w^{\beta-1,q}(\Omega))$ gives us reason to repeat the arguments.

Let $\alpha=\frac{1}{r}-1$. Then as in (11.2.6) we obtain with an appropriate choice of $\varepsilon>0$

$$\begin{aligned}
\|\mathcal{G}(\tilde{u})(t)\|_{\beta,q,w}=&\|\mathcal{G}(\tilde{u})(t)\|_{Y_w^{\beta,q}}\\
\le& c\int_0^t\Big\|\mathcal{A}^{-\alpha}e^{-(t-\tau)\mathcal{A}}\mathcal{A}^{\alpha+\frac{\beta}{2}+\frac{\varepsilon}{4}}W(u)\Big\|_{q,w}d\tau\\
\le& c\int_0^t\frac{1}{(t-\tau)^{-\alpha}}\Big\|\mathcal{A}^{\alpha+\frac{\beta}{2}+\frac{\varepsilon}{4}}W(u)\Big\|_{q,w}d\tau\\
\le& c\int_0^t\frac{1}{(t-\tau)^{-\alpha}}\|W(u)\|_{\beta+2\alpha+\varepsilon,q,w}d\tau\\
\le&\int_0^t\frac{1}{(t-\tau)^{-\alpha}}\|W(u)\|_{\rho,w}d\tau,
\end{aligned}$$

where ρ is chosen according to β as follows.

If $\beta<\frac{ns}{q}$ then we choose η_1, η_2, ρ such that

$$\frac{1}{\eta_1}=\frac{1}{q}-\frac{\beta}{ns},\quad \frac{1}{\eta_2}=\frac{1}{q}-\frac{\beta-1}{ns},\quad \frac{1}{\rho}=\frac{1}{\eta_1}+\frac{1}{\eta_2}.$$

Then one has by the restrictions on $\frac{1}{r}$

- $\rho>s$. If $n=2$ one uses the additional assumption to show this.
- $\frac{1}{\rho}+\frac{\beta+2\alpha}{ns}=\frac{2}{q}-\frac{2\beta-1}{ns}+\frac{\beta+2\alpha}{ns}<\frac{1}{q}$.
- $\beta+2\alpha<0$.

This implies with an appropriate choice of ε

$$L_w^\rho(\Omega)\hookrightarrow H_w^{\beta+2\alpha+\varepsilon,q}(\Omega),\quad H_w^{\beta,q}(\Omega)\hookrightarrow L_w^{\eta_1}(\Omega),\quad H_w^{\beta-1,q}(\Omega)\hookrightarrow L_w^{\eta_2}(\Omega).\tag{11.2.9}$$

If $\frac{ns}{q} \leq \beta < 2$, then $H_w^{\beta,q}(\Omega) \hookrightarrow L_w^{\eta_1}(\Omega)$ for every $\eta_1 \in (1,\infty)$. Then we choose ρ with $\frac{1}{q} < \frac{1}{\rho} < \frac{1}{q} - \frac{\beta+2\alpha}{ns}$, $\eta_2 > \rho$ with $\frac{1}{\eta_2} \geq \frac{1}{q} - \frac{\beta-1}{ns}$ and η_1 such that $\frac{1}{\eta_1} + \frac{1}{\eta_2} = \frac{1}{\rho}$. This implies the embeddings (11.2.9). Thus in any case we may estimate

$$\|v \cdot \nabla u\|_{\rho,w} \leq \|v\|_{\eta_1,w} \|\nabla u\|_{\eta_2,w} \leq c\|v\|_{\beta,q,w}\|u\|_{\beta,q,w} \tag{11.2.10}$$

for every $u, v \in H_w^{\beta,q}(\Omega)$. Hence we obtain as in (11.2.8)

$$\begin{aligned}\|\mathcal{G}(\tilde{u})\|_{L^r(0,T';H_w^{\beta,q})} &\leq c \left\| \int_0^t \frac{1}{(t-\tau)^{-\alpha}} (\|\tilde{u}(\tau)\|_{\beta,q,w} + \|E(\tau)\|_{\beta,q,w})^2 d\tau \right\|_{L^r(0,T')} \\ &\leq c \left(\|\tilde{u}\|_{L^r(0,T';H_w^{\beta,q})} + \|E\|_{L^r(0,T';H_w^{\beta,q})} \right)^2\end{aligned} \tag{11.2.11}$$

and

$$\begin{aligned}&\|\mathcal{G}(\tilde{u}) - \mathcal{G}(\tilde{v})\|_{L^r(0,T';H_w^{\beta,q})} \\ &\quad \leq c \left(\|E\|_{L^r(0,T';H_w^{\beta,q})} + \|\tilde{u}\|_{L^r(0,T';H_w^{\beta,q})} + \|\tilde{v}\|_{L^r(0,T';H_w^{\beta,q})} \right) \|u - v\|_{L^r(0,T';H_w^{\beta,q})}.\end{aligned}$$

Then the same iteration procedure as in the case $\beta < 1$ shows the existence of a unique fixed point $\tilde{u} = \mathcal{G}(\tilde{u})$ within a ball in $L^r(0,T'; H_w^{\beta,q}(\Omega))$ with radius δ.

We turn to the case $\beta = 2$, i.e., the case of strong solutions. One uses the estimate

$$\begin{aligned}\|\mathcal{G}(u)\|_{2,q,w} &\leq c \int_0^t \frac{1}{(t-\tau)^{-\alpha}} \|W(u)\|_{2+2\alpha+\varepsilon,q,w} d\tau \\ &\leq c \int_0^t \frac{1}{(t-\tau)^{-\alpha}} \|W(u)\|_{1,\rho,w} d\tau.\end{aligned}$$

Such a ρ can be chosen because $2 + 2\alpha = \frac{2}{r} < 1$. As above we choose ρ and η such that such that

- $\frac{1}{q} > \frac{1}{\rho} - \frac{1-\frac{2}{r}}{ns}$ to guarantee the embedding $H_w^{1,\rho}(\Omega) \hookrightarrow H_w^{\frac{2}{r}+\varepsilon,q}(\Omega)$,
- $\frac{1}{2\rho} \geq \frac{1}{q} - \frac{1}{ns}$ to obtain $H_w^{1,q}(\Omega) \hookrightarrow L_w^{2\rho}(\Omega)$.
- $\frac{1}{\eta} \geq \frac{1}{q} - \frac{2}{ns}$ which yields $H_w^{2,q}(\Omega) \hookrightarrow L_w^{\eta}(\Omega)$.

If $ns - q > 0$ the above holds if $\rho = \frac{nsq}{2ns-2q}$ and $\frac{1}{\eta} = \frac{1}{\rho} - \frac{1}{q}$. If $ns - q \leq 0$ then $H_w^{1,q}(\Omega) \hookrightarrow L_w^{2\rho}(\Omega)$ for every ρ and in addition $H_w^{2,q}(\Omega) \hookrightarrow L_w^{\eta}(\Omega)$ for every η. Then we choose any ρ with $\frac{1}{q} < \frac{1}{\rho} < \frac{1}{q} + \frac{1-\frac{1}{r}}{ns}$ and $\frac{1}{\eta} = \frac{1}{\rho} - \frac{1}{q}$ to guarantee the above.

We use this to prove $\|W(u)\|_{1,\rho,w} \leq c\|u\|_{2,q,w}^2$. To this aim we calculate

$$\begin{aligned}\|\partial_k W(u)\|_{\rho,w} &\leq \|\partial_k u \cdot \nabla u\|_{\rho,w} + \|u \cdot \partial_k \nabla u\|_{\rho,w} \\ &\leq \|\nabla u\|_{2\rho,w}^2 + \|u\|_{\eta,w} \|\nabla^2 u\|_{q,w} \leq c\|u\|_{2,q,w}^2.\end{aligned} \tag{11.2.12}$$

From now on we derive all following estimates as in the case $0 \leq \beta < 1$. This finishes the proof of existence for small data for every $\beta \in [0,2]$.

The next step is to prove the a priori estimate. Let $\tilde{u} \in B_\delta(0)$ be the fixed point of $\mathcal{G}$. Then one has by (11.2.8) and (11.2.11)

$$\begin{aligned}\|\tilde{u}\|_{L^r(0,T';H^{\beta,q}_w(\Omega))} =&\|\mathcal{G}(\tilde{u})\|_{L^r(0,T';H^{\beta,q}_w(\Omega))}\\ \leq& c\Big((\delta+K\eta)(\|\tilde{u}\|_{L^r(0,T';H^{\beta,q}_w(\Omega))}+\|E\|_{L^r(0,T';H^{\beta,q}_w(\Omega))})\\ &+\eta(\|\tilde{u}\|_{L^r(0,T';H^{\beta,q}_w(\Omega))}+\|E\|_{L^r(0,T';H^{\beta,q}_w(\Omega))})\Big).\end{aligned}$$

Choosing δ and η such that $c(\delta+2\eta)<1$ this proves

$$\|\tilde{u}\|_{L^r(0,T';H^{\beta,q}_w(\Omega))} \leq \frac{c(\delta+2\eta)}{1-c(\delta+2\eta)}\|E\|_{L^r(0,T';H^{\beta,q}_w(\Omega))}.$$

Finally, we obtain for $\beta<1$

$$\begin{aligned}\|u\|_{L^r(0,T';H^{\beta,q}_w(\Omega))} \leq&\|\tilde{u}\|_{L^r(0,T';H^{\beta,q}_w(\Omega))}+\|E\|_{L^r(0,T';H^{\beta,q}_w(\Omega))}\\ \leq& c\|E\|_{L^r(0,T';H^{\beta,q}_w(\Omega))}\\ \leq& c\Big(\Big(\int_0^{T'}\|e^{-\tau\mathcal{A}}u_0\|^r_{\beta,q,w}d\tau\Big)^{\frac{1}{r}}+\|f\|_{L^r(0,T',Y^{\beta-2}_w(\Omega))}\\ &+\|k\|_{L^r(0,T';L^\mu_w)\cap H_0^{\frac{\beta}{2},r}((0,T'];W^{-1,q}_{w,0})}+\|g\|_{L^r(0,T';T^{\beta,q}_w)\cap H_0^{\frac{\beta}{2},r}((0,T'];T^{0,q}_w)}\Big)\end{aligned}$$

by the a priori estimate in the linear case in Theorem 9.5.4. If $\beta\geq 1$ one obtains the estimate

$$\begin{aligned}\|u\|_{L^r(0,T';H^{\beta,q}_w(\Omega))} \leq& c\Big(\Big(\int_0^{T'}\|e^{-\tau\mathcal{A}}u_0\|^r_{\beta,q,w}d\tau\Big)^{\frac{1}{r}}+\|f\|_{L^r(0,T',Y^{\beta-2}_w(\Omega))}\\ &+\|k\|_{L^r(0,T';H^{\beta-1,q}_w)\cap H_0^{\frac{\beta}{2},r}((0,T'];W^{-1,q}_{w,0})}+\|g\|_{L^r(0,T';T^{\beta,q}_w)\cap H_0^{\frac{\beta}{2},r}((0,T'];T^{0,q}_w)}\Big)\end{aligned}$$

analogously.

Since u is a very weak solution to the instationary Stokes problem

$$\partial_t u-\Delta u+\nabla p=f-W(u), \tag{11.2.13}$$

we get the estimate (11.2.4) from the linear case. More precisely let $T''\in(0,T']$ with $T''<\infty$ and choose ρ as in the estimates (11.2.7), (11.2.10), (11.2.12). Then by Hölder's inequality we can estimate in the case $\beta<1$

$$\begin{aligned}&\|\partial_t u|_{Y^{2,q'}_{w',\sigma}(\Omega)}\|_{L^{\frac{r}{2}}(0,T'';Y^{\beta-2,q}_{w,\sigma}(\Omega))}\\ \leq& c\left(\|f\|_{L^{\frac{r}{2}}(0,T'';Y^{\beta-2,q}_w(\Omega))}+\|W(u)\|_{L^{\frac{r}{2}}(0,T'';Y^{\beta-2,q}_{w,\sigma}(\Omega))}\right)\\ \leq& c(T'')\left(\|f\|_{L^r(0,T';Y^{\beta-2,q}_w(\Omega))}+\|W(u)\|_{L^{\frac{r}{2}}(0,T';H^{-1,\rho}_w(\Omega))}\right)\\ \leq& c\Big(\|f\|_{L^r(0,T';Y^{\beta-2,q}_w(\Omega))}+\|u\|^2_{L^r(0,T';H^{\beta,q}_w(\Omega))}\\ &+\|k\|_{L^r(0,T';L^\mu_w(\Omega))}\|u\|_{L^r(0,T';H^{\beta,q}_w(\Omega))}\Big)\\ \leq& c\|f\|_{L^r(0,T';Y^{\beta-2,q}_w(\Omega))}+c(\eta+\delta)\|u\|_{L^r(0,T';H^{\beta,q}_w(\Omega))}.\end{aligned}$$

If $\beta \in [1,2)$ one estimates analogously

$$\begin{aligned}\|\partial_t u|_{Y^{2,q'}_{w',\sigma}(\Omega)}\|_{L^{\frac{r}{2}}(0,T'';Y^{\beta-2}_{w,\sigma}(\Omega))} &\leq c(T'')(\|f\|_{L^r(0,T';Y^{\beta-2,q}_w(\Omega))} + \|W(u)\|_{L^{\frac{r}{2}}(0,T';L^{\rho}_{w,\sigma}(\Omega))})\\ &\leq c\|f\|_{L^r(0,T';Y^{\beta-2,q}_w(\Omega))} + c(\eta+\delta)\|u\|_{L^r(0,T';H^{\beta,q}_w(\Omega))}\end{aligned}$$

and for $\beta = 2$

$$\begin{aligned}\|P_{q,w}\partial_t u\|_{L^{\frac{r}{2}}(0,T'';L^q_w(\Omega)} &\leq c(T'')\left(\|f\|_{L^r(0,T';L^q_w(\Omega))} + \|W(u)\|_{L^{\frac{r}{2}}(0,T';W^{1,\rho}_w(\Omega))}\right)\\ &\leq c\|f\|_{L^r(0,T';L^q_w(\Omega))} + c(\eta+\delta)\|u\|_{L^r(0,T';H^{2,q}_w(\Omega))}.\end{aligned}$$

Note that the equation (11.2.13) is only tested with functions in $Y^{2,q'}_{w',\sigma}(\Omega)$ and only holds in this sense. Thus the distributional derivative $\partial_t u$ may contain a gradient part which is not a function in time. Then the estimate for u proves (11.2.4).

Uniqueness can be proved in the same way as in [17]: Let $v \in L^r(0,T';H^{\beta,q}_w(\Omega))$ be a very weak solution corresponding to the same data f, k, g and u_0. Then $U := u - v$ solves

$$\begin{aligned}&\partial_t U - \Delta U + \nabla P = -\mathrm{div}\,(Uu) - \mathrm{div}\,(vU) + kU,\\ &\mathrm{div}\, U = 0, \quad U|_{\partial\Omega} = 0, \quad U(0) = 0\end{aligned}$$

in the very weak sense. Then for $\beta < 1$ one obtains as above

$$\begin{aligned}\|U\|_{L^r(0,T';H^{\beta,q}_w(\Omega))} \leq & c\left(\|u\|_{L^r(0,T';H^{\beta,q}_w)} + \|v\|_{L^r(0,T';H^{\beta,q}_w)}\right.\\ &\left. + \|k\|_{L^r(0,T';L^{\mu}_w(\Omega))}\right)\|U\|_{L^r(0,T';H^{\beta,q}_w(\Omega))}\end{aligned}$$

with a constant c that is independent of T'. A corresponding inequality holds in the case $\beta \geq 1$. In particular it holds for T' replaced by any $T''' \in (0,T']$. If T''' is sufficiently small such that

$$\|u\|_{L^r(0,T''';H^{\beta,q}_w(\Omega))} + \|v\|_{L^r(0,T''';H^{\beta,q}_w(\Omega))} + \|k\|_{L^r(0,T''';L^{\mu}_w(\Omega))} < \frac{1}{2c},$$

we obtain $\|U\|_{L^r(0,T''';H^{\beta,q}_w(\Omega))} \leq 0$ or $U = 0$ on $[0,T''')$. If $T''' < T'$ we assume that T''' is maximal with the property $u = v$ on $[0,T''')$. However, then we may repeat this procedure and obtain $u = v$ on a bigger interval. This is a contradiction. Thus u is unique in $L^r(0,T';H^{\beta,q}_w(\Omega))$ and the proof is complete. □

Remark 11.2.4. Choose β, r, q according to Theorem 11.2.3.

We now prove that in this case the solution $u \in L^r(0,T;H^{\beta,q}_w(\Omega))$ fulfills Serrin's condition [43] in the sense that $u \in L^r(0,T',L^{\eta}(\Omega))$, where $\frac{1}{r} + \frac{n}{\eta} < 1$.

If $ns - q\beta > 0$ then for the number ρ that fulfills $\frac{1}{2\rho} = \frac{1}{q} - \frac{\beta}{ns}$ one has by Lemma 10.1.4 and Lemma 3.2.2

$$H^{\beta,q}_w(\Omega) \hookrightarrow L^{2\rho}_w(\Omega) \hookrightarrow L^{\frac{2\rho}{s}}(\Omega)$$

and

$$\frac{2}{r} + \frac{n}{\frac{2\rho}{s}} < -\frac{ns}{q} + \beta + 1 + \frac{ns - q\beta}{q} = 1.$$

If $ns - q\beta \leq 0$ then $H_w^{\beta,q}(\Omega) \hookrightarrow L^\eta(\Omega)$ for every $\eta \in (1, \infty)$. Since $r > 2$, this η can be chosen such that $\frac{2}{r} + \frac{n}{\eta} < 1$.

The reason why there appears "$<$" instead of "$\leq$" as in the unweighted case [43], [19], is that the boundedness of imaginary powers is not proved for the Stokes operator in spaces weighted with arbitrary Muckenhoupt weights. Thus we have to work without an exact characterization of the domains of fractional powers of the Stokes operator $\mathcal{D}(\mathcal{A}^\alpha)$ and use the embedding (11.2.1) instead.

A Appendix

A.1 The Laplace Equation in a Bounded Domain

The aim of this section is to prove Theorem 4.1.3

We begin proving the solvability of the Laplace equation in bounded domains. Since the whole space and the half space case has been shown in [25], we start with the bent half space. For a Lipschitz continuous function $\sigma : \mathbb{R}^{n-1} \to \mathbb{R}$ we define the bent half space by

$$H_\sigma := \{x = (x', x_n) \in \mathbb{R}^n \mid x_n > \sigma(x')\}.$$

The proof follows the ideas of [20], [45] and [46].

It can also be shown that the constants are independent of λ. However, since this fact is not needed in the rest of this thesis, it is not worked out here.

Theorem A.1.1. *Let $n \geq 2$, $1 < q < \infty$, $w \in A_q$, $0 < \varepsilon < \frac{\pi}{2}$ and let $\sigma \in C^{1,1}(\mathbb{R}^{n-1})$ be bounded. Then there exist constants $K = K(n, q, w, \varepsilon) > 0$ and $\lambda_0 = \lambda_0(\sigma, n, q, w, \varepsilon) > 0$ such that: whenever $\|\nabla\sigma\|_\infty < K$, then for every $f \in L^q_w(H_\sigma)$ and all $\lambda \in \Sigma_\varepsilon$ with $|\lambda| > \lambda_0$ there exists a unique solution $u \in W^{2,q}_w(\Omega)$ of the Laplace-resolvent problem*

$$\lambda u - \Delta u = f \quad \text{in } H_\sigma \text{ and } u|_{\partial H_\sigma} = 0.$$

Moreover, one has the estimate

$$|\lambda| \|u\|_{q,w} + \|\nabla^2 u\|_{q,w} \leq c\|f\|_{q,w}$$

with $c = c(\sigma, q, w, \lambda)$.

Proof. We use the coordinate transformation $\phi : H_\sigma \to \mathbb{R}^n_+$ given by $\tilde{x} = (\tilde{x}', \tilde{x}_n) = \phi(x) = (x', x_n - \sigma(x'))$. Obviously ϕ is a bijection with functional determinant equal to 1. For a function u on H_σ we define the transformed function $\tilde{u}(\tilde{x}) = u(\phi^{-1}\tilde{x})$, $\tilde{x} \in \mathbb{R}^n_+$. By $\tilde{\partial}_j, \tilde{\nabla}, ...$ we denote the derivatives with respect to the variable $\tilde{x} \in \mathbb{R}^n_+$. In addition we set $\tilde{w} = w \circ \phi^{-1}$.

Then one has

$$\begin{aligned} \lambda u(x) - \Delta u(x) &= \lambda \tilde{u}(\tilde{x}) - \tilde{\Delta}\tilde{u}(\tilde{x}) + 2(\nabla'\sigma(x), 0)\tilde{\nabla}\tilde{\partial}_n\tilde{u}(\tilde{x}) \\ &\quad + \Delta'\sigma(x)\tilde{\partial}_n\tilde{u}(\tilde{x}) - |\nabla'\sigma(x)|^2\tilde{\partial}_n^2\tilde{u}(\tilde{x}) \\ &= (\lambda - \tilde{\Delta})\tilde{u}(\tilde{x}) + R\tilde{u}(\tilde{x}) \end{aligned}$$

with

$$\|R\tilde{u}\|_{q,\tilde{w},\mathbb{R}^n_+} \leq 2\|\nabla'\sigma\|_\infty \|\tilde{\nabla}^2\tilde{u}\|_{q,\tilde{w},\mathbb{R}^n_+} + \|\Delta'\sigma\|_\infty \|\tilde{\partial}_n\tilde{u}\|_{q,\tilde{w},\mathbb{R}^n_+} + \|\nabla'\sigma\|^2_\infty \|\tilde{\nabla}^2\tilde{u}\|_{q,\tilde{w},\mathbb{R}^n_+}.$$

One can estimate using a weighted version of Ehrling's inequality [25]

$$\|\Delta'\sigma\|_\infty \|\tilde{\partial}_n \tilde{u}\|_{q,\tilde{w},\mathbb{R}^n_+} \leq c\delta(\lambda_0 \|\tilde{u}\|_{q,\tilde{w},\mathbb{R}^n_+} + \|\tilde{\nabla}^2 \tilde{u}\|_{q,\tilde{w},\mathbb{R}^n_+}),$$

for every $\delta \in (0,1)$ and $\lambda_0 = \lambda_0(\delta) = \|\Delta'\sigma\|^2_\infty \delta^{-2}$. Thus using the a priori estimates of solutions to the Laplace-resolvent problem in the half space we obtain the estimate

$$\begin{aligned}\|Ru\|_{q,\tilde{w},\mathbb{R}^n_+} &\leq (2\|\nabla'\sigma\|_\infty + \|\nabla'\sigma\|^2_\infty + c\delta)\|\tilde{\nabla}^2\tilde{u}\|_{q,\tilde{w},\mathbb{R}^n_+} + c\delta\|\lambda\tilde{u}\|_{q,\tilde{w},\mathbb{R}^n_+} \\ &\leq C(\delta,\sigma)(\|\tilde{\nabla}^2\tilde{u}\|_{q,\tilde{w},\mathbb{R}^n_+} + \|\lambda\tilde{u}\|_{q,\tilde{w},\mathbb{R}^n_+}) \\ &\leq C(\delta,\sigma)c(\|\tilde{\Delta}\tilde{u} + \lambda\tilde{u}\|_{q,\tilde{w},\mathbb{R}^n_+}).\end{aligned}$$

Since the constant c in the a priori estimate for the Laplace-resolvent equation is independent of $\lambda \in \Sigma_\varepsilon$, the smallness of δ and σ implies the smallness of the constant $C(\delta,\sigma)c$.

We define the function space

$$Y^{2,q}_{\tilde{w},\lambda}(\mathbb{R}^n_+) := \left\{u \in W^{2,q}_{\tilde{w}}(\mathbb{R}^n_+) \mid u|_{\mathbb{R}^{n-1}} = 0\right\}$$

equipped with the norm

$$\|u\|_{Y^{2,q}_{q,\lambda}(\mathbb{R}^n_+)} = |\lambda|\|u\|_{q,\tilde{w},\mathbb{R}^n_+} + \|\nabla^2 u\|_{q,\tilde{w},\mathbb{R}^n_+}.$$

Then by Theorem 4.1.1 the operator

$$(\lambda - \tilde{\Delta}) : Y^{2,q}_{\tilde{w},\lambda}(\mathbb{R}^n_+) \to L^q_{\tilde{w}}(\mathbb{R}^n_+)$$

is invertible and we obtain from [36, IV. Theorem 1.16], choosing σ and δ such that $C(\delta,\sigma)c < 1$, that the perturbed operator $(\lambda - \tilde{\Delta}) + R$ is invertible as well with

$$\|(\lambda - \tilde{\Delta} + R)^{-1}\|_{\mathcal{L}(L^q_{\tilde{w}}(\mathbb{R}^n_+), Y^{2,q}_{\tilde{w},\lambda}(\mathbb{R}^n_+)))} \leq \frac{1}{1 - C(\delta,\sigma)c}\|(\lambda - \tilde{\Delta})^{-1}\|_{\mathcal{L}(L^q_{\tilde{w}}(\mathbb{R}^n_+), Y^{2,q}_{\tilde{w},\lambda}(\mathbb{R}_+))}.$$

Thus for $\tilde{f} := f \circ \phi$ there exists a solution $\tilde{u} \in Y^{2,q}_{\tilde{w},\lambda}(\mathbb{R}^n_+)$ of $(\lambda - \tilde{\Delta} + R)\tilde{u} = \tilde{f}$ with $\|\lambda\tilde{u}\|_{q,\tilde{w},\mathbb{R}^n_+} + \|\tilde{u}\|_{2,q,\tilde{w},\mathbb{R}^n_+} \leq c\|\tilde{f}\|_{q,\tilde{w},\mathbb{R}^n_+}$. Hence $u := \tilde{u} \circ \phi^{-1}$ solves $(\lambda - \Delta)u = f$ and fulfills

$$\|\lambda u\|_{q,w,H_\sigma} + \|\nabla^2 u\|_{q,w,H_\sigma} \leq c(\|\lambda\tilde{u}\|_{q,\tilde{w},\mathbb{R}^n_+} + \|\tilde{\nabla}^2\tilde{u}\|_{q,\tilde{w},\mathbb{R}^n_+}) \leq c\|\tilde{f}\|_{q,\tilde{w},\mathbb{R}^n_+} \leq c\|f\|_{q,w,H_\sigma}.$$

This finishes the proof. □

For a bounded $C^{1,1}$-domain Ω the domain of the Dirichlet-Laplacian is given by

$$Y^{2,q}_w(\Omega) := \{u \in W^{2,q}_w(\Omega) \mid u|_{\partial\Omega} = 0\}.$$

Lemma A.1.2. *Let $1 < q < \infty$, $w \in A_q$, $\lambda \in \Sigma_\varepsilon \cup \{0\}$ and let Ω be a bounded $C^{1,1}$-domain. Then*

1. *$(\lambda - \Delta) : Y^{2,q}_w \to L^q_w(\Omega)$ is injective.*
2. *$(\lambda - \Delta)(Y^{2,q}_w)$ is dense in $L^q_w(\Omega)$.*

Proof. 1. By (3.2.2) there exists an s, $1 < s < \infty$, such that $L^q_w(\Omega) \subset L^s(\Omega)$. Since $(\lambda - \Delta)$ is injective on $Y^{2,s}_1(\Omega)$, the same is true on $Y^{2,q}_w(\Omega)$.
2. Again by (3.2.2) we can choose $1 < r < \infty$ such that $L^r(\Omega) \subset L^q_w(\Omega)$. Then

$$(\lambda - \Delta)(Y^{2,q}_w(\Omega)) \supset (\lambda - \Delta)(Y^{2,r}_1(\Omega)) = L^r(\Omega)$$

Since the embedding $L^r(\Omega) \hookrightarrow L^q_w(\Omega)$ is dense, the proof is finished. □

Lemma A.1.3. *Let Ω be a bounded $C^{1,1}$-domain, $1 < q < \infty$, $w \in A_q$, $f \in L^q_w(\Omega)$, $0 < \varepsilon < \frac{\pi}{2}$, $\lambda \in \Sigma_\varepsilon \cup \{0\}$ and let $u \in W^{2,q}_w(\Omega)$ be a solution of*

$$(\lambda - \Delta)u = f \quad \textit{and} \quad u|_{\partial\Omega} = 0. \tag{A.1.1}$$

Then u fulfills the estimate

$$|\lambda| \|u\|_{q,w} + \|u\|_{2,q,w} \le c\|f\|_{q,w}, \tag{A.1.2}$$

where $c = c(\Omega, q, w, \lambda)$

Proof. We first prove the weaker estimate

$$|\lambda| \|u\|_{q,w} + \|u\|_{2,q,w} \le c(\|f\|_{q,w} + \|u\|_{1,q,w}). \tag{A.1.3}$$

To this aim we cover $\overline{\Omega}$ by finitely many balls $\{B_j\}_j$ such that $\partial\Omega \cap B_j$ can be described by the graph of a $C^{1,1}$-function σ_j, such that, extended appropriately to a function on $\mathbb{R}^{n-1}$, it fulfills the assumptions of Theorem A.1.1. Let $\{\psi_n\}$ be a decomposition of unity subordinate to the covering $\{B_j \cap \overline{\Omega}\}_j$ of $\overline{\Omega}$.

Multiplying ψ_j and adding $\lambda_0 \psi_j u$ to both sides of (A.1.1) we obtain

$$(\lambda + \lambda_0)(\psi_j u) - \Delta(\psi_j u) = f_j + \lambda_0 \psi_j u,$$

where $\lambda_0 > 0$ is chosen large enough such that the assumptions of Theorem A.1.1 are fulfilled and with

$$f_j := \psi_j f - \Delta\psi_j \cdot u - 2\nabla\psi_j \nabla u.$$

Then f_j fulfills the estimate

$$\|f_j\|_{q,w} \le c(\psi_j)(\|f\|_{q,w} + \|u\|_{1,q,w})$$

and we obtain from Theorem A.1.1

$$|\lambda + \lambda_0| \|\psi_j u\|_{q,w} + \|\nabla^2(\psi_j u)\|_{q,w} \le c(\psi_j, \lambda_0)(\|f\|_{q,w} + \|u\|_{1,q,w}).$$

Summing over j we obtain (A.1.3).

We have to get rid of the last term in (A.1.3). Assume that (A.1.2) is not true. Then there exist two sequences $(u_j)_j \subset W^{2,q}_w(\Omega)$ and $(f_j)_j \subset L^q_w(\Omega)$ with $\lambda u_j - \Delta u_j = f_j$, $u_j|_{\partial\Omega} = 0$ and

$$|\lambda| \|u_j\|_{q,w} + \|u_j\|_{2,q,w} > j\|f_j\|_{q,w}.$$

Without loss of generality we may assume that

$$|\lambda| \|u_j\|_{q,w} + \|u_j\|_{2,q,w} = 1 \text{ and } \|f_j\|_{q,w} \xrightarrow{j\to\infty} 0.$$

Turning to a subsequence we may assume that

$$u_j \rightharpoonup u \;\text{ in } W_w^{2,q}(\Omega).$$

By the compact embedding $W_w^{2,q}(\Omega) \hookrightarrow W_w^{1,q}(\Omega)$ in Lemma 3.3.5 one has $u_j \xrightarrow{j\to\infty} u$ in $W_w^{1,q}(\Omega)$. Therefore,

$$\lambda u - \Delta u \xleftarrow{W_w^{-1,q}(\Omega)} \lambda u_j - \Delta u_j = f_j \xrightarrow{L_w^q(\Omega)} 0$$

From the injectivity of $(\lambda - \Delta)$ shown in Lemma A.1.2 we obtain $u = 0$. Hence using the weaker estimate above we obtain

$$|\lambda| \|u_j\|_{q,w} + \|u_j\|_{2,q,w} \le c(\|f_j\|_{q,w} + \|u_j\|_{1,q,w}) \xrightarrow{j\to\infty} 0.$$

This is a contradiction, and the proof is finished. □

Proof of Theorem 4.1.3. Approximate f by a sequence of functions

$$(f_n)_n \subset (\lambda - \Delta)(Y_w^{2,q}(\Omega)).$$

This is possible by Lemma A.1.2. By Lemma A.1.3 the sequence (u_n) of solutions with respect to f_n is a Cauchy sequence converging to some $u \in Y_w^{2,q}(\Omega)$. This function u solves our problem. Uniqueness and apriori estimate follow from Lemma A.1.3. □

A.2 Regularity of the Helmholtz Decomposition

The aim of this section is to prove Theorem 4.4.1. The proof follows the lines of [45] where the continuity Helmholtz projection in L^q-spaces is proven.

We also use the corresponding results and auxiliary results in weighted spaces proven in [24].

Since a part of the proof takes place in the unbounded bent half space, we introduce homogeneous Sobolev spaces. For $k \in \mathbb{N}_0$ and a domain Ω we set

$$\widehat{W}_w^{k,q}(\Omega) = \{W_{loc}^{k,1}(\overline{\Omega}) \mid \partial^\alpha u \in L_w^q(\Omega),\ |\alpha| = k\}$$

equipped with the semi-norm

$$\|u\|_{\widehat{W}_w^{k,q}(\Omega)} := \|\nabla^k u\|_{L_w^q(\Omega)}.$$

If one considers the cosets with respect to polynomials of degree $\le k-1$, $\|\cdot\|_{\widehat{W}_w^{k,q}(\Omega)}$ is a norm and $\widehat{W}_w^{k,q}(\Omega)$ is a Banach space.

Moreover, we set

$$\widehat{W}_{w,0}^{-k,q}(\Omega) := \left(\widehat{W}_{w'}^{k,q'}(\Omega)\right)'.$$

Similarly to the spaces $W_{w,0}^{-k,q}(\Omega)$ we denote the norm in $\widehat{W}_{w,0}^{-k,q}(\Omega)$ by $\|\cdot\|_{-k,q,w,0}$.

Lemma A.2.1. *Let $k \in \{0,1,2\}$ and for $\sigma \in C^{k,1}(\mathbb{R}^{n-1})$ let $H_\sigma = \{x \in \mathbb{R}^n \mid x_n > \sigma(x')\}$ be the bent half space.*

Then there exists $K = K(q,w)$ such that if $\|\sigma\|_{C^{k,1}(\mathbb{R}^{n-1})} < K$ one has

$$\|\nabla p\|_{k,q,w} \leq c \sup_{\phi \in \widehat{W}^{1,q'}_{w'}(H_\sigma)} \frac{|\langle \nabla p, \nabla \phi\rangle|}{\|\nabla \phi\|_{-k,q',w',0}} \quad \text{for every} \quad p \in W^{k+1,q}_w(H_\sigma)$$

for some constant $c = c(k,q,w,\sigma) > 0$. The supremum is taken over all ϕ with $\|\nabla\phi\|_{-k,q',w',0} \neq 0$.

Note that the functions $\phi \in \widehat{W}^{1,q'}_{w'}(H_\sigma)$ are smooth enough to take the gradient in the usual weak sense.

Proof. The case $k=0$ has been shown in [24] for a C^1-function σ. However, the same proof can be used if σ is only of class $C^{0,1}$.

We start with the half space case $H_\sigma = \mathbb{R}^n_+$. Let $p \in W^{k+1,q}_w(\mathbb{R}^n_+)$. Then for $j = 1,...,n-1$ we can calculate using the case $k=0$ and the fact that $C^\infty_0(\overline{\mathbb{R}^n_+})$ is dense in $\widehat{W}^{1,q'}_{w'}(\mathbb{R}^n_+)$ shown in [25]

$$\begin{aligned}
\|\partial_j \nabla p\|_{q,w} &\leq c \sup_{\phi \in \widehat{W}^{1,q'}_{w'}(\mathbb{R}^n_+)} \frac{|\langle \nabla \partial_j p, \nabla\phi\rangle|}{\|\nabla\phi\|_{q',w'}} = c \sup_{\phi \in C^\infty_0(\overline{\mathbb{R}^n_+})} \frac{|\langle \nabla p, \nabla \partial_j \phi\rangle|}{\|\nabla\phi\|_{q',w'}} \\
&\leq c \sup_{\phi \in C^\infty_0(\overline{\mathbb{R}^n_+})} \frac{|\langle \nabla p, \nabla \partial_j \phi\rangle|}{\|\nabla\partial_j\phi\|_{-1,q',w',0}} \leq c \sup_{\phi \in \widehat{W}^{1,q'}_{w'}(\mathbb{R}^n_+)} \frac{|\langle \nabla p, \nabla\phi\rangle|}{\|\nabla\phi\|_{-1,q',w',0}},
\end{aligned}$$

where we have used the estimate $\|\partial_j \nabla\phi\|_{-1,q',w',0} \leq c\|\nabla\phi\|_{q',w'}$ which holds true for $j = 1,...,n-1$ and $\phi \in C^\infty_0(\overline{\mathbb{R}^n_+})$ since

$$|\langle \nabla\partial_j\phi, \zeta\rangle_{\mathbb{R}^n_+}| = |\langle \nabla\phi, \partial_j\zeta\rangle_{\mathbb{R}^n_+}| \leq \|\nabla\phi\|_{q',w'}\|\zeta\|_{1,q,w} \quad \text{for every} \quad \zeta \in \widehat{W}^{1,q}_w(\mathbb{R}^n_+).$$

For the derivative with respect to x_n one has

$$\begin{aligned}
\|\partial_n \nabla p\|_{q,w} &\leq \|\partial_{n,n} p\|_{q,w} + \sum_{j=1}^{n-1} \|\partial_j \partial_n p\|_{q,w} \\
&\leq \|\operatorname{div} \nabla p\|_{q,w} + \sum_{j=1}^{n-1} \|\partial_j \nabla p\|_{q,w} \\
&\leq c \left(\sup_{\phi \in C^\infty_0(\mathbb{R}^n_+)} \frac{|\langle \nabla p, \nabla\phi\rangle|}{\|\phi\|_{q',w'}} + \sup_{\phi \in \widehat{W}^{1,q'}_{w'}(\mathbb{R}^n_+)} \frac{|\langle \nabla p, \nabla\phi\rangle|}{\|\nabla\phi\|_{-1,q',w',0}} \right) \\
&\leq c \sup_{\phi \in \widehat{W}^{1,q'}_{w'}(\mathbb{R}^n_+)} \frac{|\langle \nabla p, \nabla\phi\rangle|}{\|\nabla\phi\|_{-1,q',w',0}},
\end{aligned}$$

where the third estimate holds by the density of $C^\infty_0(\mathbb{R}^n_+)$ in $L^{q'}_{w'}(\mathbb{R}^n_+)$ and the last estimate holds since for $\phi \in C^\infty_0(\mathbb{R}^n_+)$ one has

$$|\langle \nabla\phi, \eta\rangle| = |\langle \phi, \operatorname{div}\eta\rangle| \leq \|\phi\|_{q',w'}\|\eta\|_{1,q,w} \quad \Rightarrow \quad \|\nabla\phi\|_{-1,q',w',0} \leq \|\phi\|_{q',w'}.$$

Together with the estimate for $k = 0$ we obtain the estimate for $k = 1$. To get the estimate for $k = 2$ we repeat the above arguments replacing the estimate for $k = 0$ by the one for $k = 1$.

We turn to the case $\sigma \neq 0$ and define the coordinate transformation

$$\psi : H_\sigma \to \mathbb{R}^n_+, \quad \psi(x', x_n) = (x', x_n - \sigma(x')).$$

Let $\tilde{p} : \mathbb{R}^n_+ \to \mathbb{R}$, $\tilde{p}(\tilde{x}) = p(\psi^{-1}(\tilde{x}))$, $\tilde{w}(\tilde{x}) = w(\psi^{-1}(\tilde{x}))$ and $\tilde{\partial}_j, \tilde{\nabla}, ...$ be the derivatives with respect to $\tilde{x}$. Then one has

$$\partial_j p(x) = (\tilde{\partial}_j - (\partial_j\sigma)\tilde{\partial}_n)\tilde{p}(\tilde{x}) \;\text{ and }\; \tilde{\partial}_j\tilde{p}(\tilde{x}) = (\partial_j + (\partial_j\sigma)\partial_n)p(x). \tag{A.2.1}$$

Hence we get the estimate

$$\|\tilde{\nabla}\tilde{p}\|_{k,q,\tilde{w},\mathbb{R}^n_+} \leq c(1 + \|\sigma\|_{C^{k,1}})^{k+1}\|\nabla p\|_{k,q,w,H_\sigma} \tag{A.2.2}$$

and for $\eta \in \widehat{W}^{1,q'}_{w'}(H_\sigma)$

$$\begin{aligned}
\|\nabla\eta\|_{-k,q',w',0,H_\sigma} &= \sup_{\phi\in\widehat{W}^{k,q}_w(H_\sigma)} \frac{|\langle\nabla\eta,\phi\rangle_{H_\sigma}|}{\|\nabla^k\phi\|_{q,w,H_\sigma}}\\
&\leq c(1 + \|\sigma\|_{C^{k,1}})^k \sup_{\tilde{\phi}\in\widehat{W}^{k,q}_{\tilde{w}}(\mathbb{R}^n_+)} \frac{|\langle(\tilde{\nabla} - (\nabla'\sigma)\tilde{\partial}_n)\tilde{\eta},\tilde{\phi}\rangle_{\mathbb{R}^n_+}|}{\|\tilde{\nabla}\tilde{\phi}\|_{q,\tilde{w},\mathbb{R}^n_+}}\\
&\leq c(1 + \|\sigma\|_{C^{k,1}})^k(\|\tilde{\nabla}\tilde{\eta}\|_{-k,q,\tilde{w},0,\mathbb{R}^n_+} + \|\tilde{\nabla}\tilde{\eta}\|_{-k,q,\tilde{w},0,\mathbb{R}^n_+}\|\sigma\|_{C^{k,1}})\\
&= c(1 + \|\sigma\|_{C^{k,1}})^{k+1}\|\tilde{\nabla}\tilde{\eta}\|_{-k,q,\tilde{w},0,\mathbb{R}^n_+}.
\end{aligned}$$

By (A.2.1) and (A.2.2) an elementary calculation yields

$$|\langle\nabla p,\nabla\eta\rangle_{H^\sigma}| \geq |\langle\tilde{\nabla}\tilde{p},\tilde{\nabla}\tilde{\eta}\rangle_{\mathbb{R}^n_+}| - c\|\sigma\|_{C^{k,1}}(1 + \|\sigma\|_{C^{k,1}})\|\tilde{\nabla}\tilde{p}\|_{k,q,\tilde{w},\mathbb{R}^n_+}\|\tilde{\nabla}\tilde{\eta}\|_{-k,q',\tilde{w}',\mathbb{R}^n_+,0}$$

and we can estimate

$$\begin{aligned}
&\sup_{\phi\in\widehat{W}^{1,q'}_{w'}(H_\sigma)} \frac{|\langle\nabla p,\nabla\phi\rangle_{H_\sigma}|}{\|\nabla\phi\|_{-k,q',w',0,H_\sigma}}\\
&\qquad\geq c(1 + \|\sigma\|_{C^{k,1}})^{-(k+1)}\Big(\sup_{\tilde{\phi}\in\widehat{W}^{1,q'}_{\tilde{w}'}(\mathbb{R}^n_+)} \frac{|\langle\tilde{\nabla}\tilde{p},\tilde{\nabla}\tilde{\phi}\rangle_{R^n_+}}{\|\tilde{\nabla}\tilde{\phi}\|_{-k,q',\tilde{w}',0,\mathbb{R}^n_+}}\\
&\qquad\quad - c\|\sigma\|_{C^{k,1}}(1 + \|\sigma\|_{C^{k,1}})\|\tilde{\nabla}\tilde{p}\|_{k,q,\tilde{w},\mathbb{R}^n_+}\Big)\\
&\qquad\geq (c_1(1 + \|\sigma\|_{C^{k,1}})^{-2(k+1)} - c_2\|\sigma\|_{C^{k,1}}(1 + \|\sigma\|_{C^{k,1}}))\|\nabla p\|_{k,q,w,H_\sigma}.
\end{aligned}$$

Now we can choose $\|\sigma\|_{C^{2,1}} < K$ small enough so that

$$c_1(1 + \|\sigma\|_{C^{k,1}})^{-2(k+1)} - c_2\|\sigma\|_{C^{k,1}}(1 + \|\sigma\|_{C^{k,1}})$$

is a positive constant. □

Lemma A.2.2. *Let $k = 0, 1$ and let $\phi \in \widehat{W}^{1,q'}_{w'}(H_\sigma)$ satisfy $\int_{H_\sigma} \phi = 0$ and* $\operatorname{supp} \phi \subset B_r$ *for some ball B_r with radius r. Then one has*

$$\|\phi\|_{-k,q,w,0} \leq c\|\nabla\phi\|_{-k-1,q,w,0}.$$

Proof. Let $\eta \in W^{k,q'}_{w'}(H_\sigma)$ with $\int \eta = 0$. Then there exists $\zeta \in W^{k+1,q'}_{w'}(H_\sigma)$ such that

$$\operatorname{div} \zeta = \eta, \quad \zeta|_{\partial H_\sigma} = 0 \quad \text{and} \quad \|\zeta\|_{k+1,q',w'} \leq c\|\eta\|_{k,q',w'}.$$

If $k = 0$ this is possible by Theorem 4.3.1 with $\Omega = H_\sigma \cap B_r$, and if $k = 1$ let ζ be the strong solution to a Stokes resolvent problem in the bent half space with vanishing force and boundary condition and with divergence η which exists by [26].

Then we may estimate

$$|\langle \phi, \eta\rangle_{H_\sigma}| = |\langle \phi, \operatorname{div} \zeta\rangle_{H_\sigma}| \leq \|\nabla\phi\|_{-k-1,q,w,0}\|\zeta\|_{k+1,q',w'} \leq c\|\nabla\phi\|_{-k-1,q,w,0}\|\eta\|_{k,q',w'}.$$

This proves the assertion. □

Proof of Theorem 4.4.1. For $u \in W^{k,q}_w(\Omega)$ one has $P_{q,w}u = u - \nabla p$, where p is the solution to the weak Neumann problem

$$\langle \nabla p, \nabla \phi\rangle = \langle u, \nabla\phi\rangle \quad \text{for every} \quad \phi \in \widehat{W}^{1,q'}_{w'}(\Omega).$$

Choose p such that $\int_\Omega p = 0$. Let $\{G_j\}_{j=1}^m$ be a covering of Ω such that $\partial\Omega \cap G_j$ is the graph of a $C^{k,1}$-function α_j with sufficiently small norm such that we can apply Lemma A.2.1. We may assume that α_j is extended to a function $\alpha_j : \mathbb{R}^{n-1} \to \mathbb{R}$, also with sufficiently small $C^{k,1}$-norm. Let $(\psi_j)_{j=1}^m$ be a decomposition of unity subordinate to the covering $\{G_j\}_{j=1}^m$.

Then for fixed j one has $\psi_j p \in W^{1,q}_w(H_{\alpha_j})$ and

$$\begin{aligned}\langle \nabla(\psi_j p), \nabla\phi\rangle_\Omega &= \langle p\nabla\psi_j, \nabla\phi\rangle_\Omega + \langle \nabla p, \nabla(\phi\psi_j)\rangle_\Omega - \langle \nabla p, (\nabla\psi_j)\phi\rangle_\Omega \\ &= \langle p\nabla\psi_j, \nabla\phi\rangle_\Omega + \langle u, \nabla(\phi\psi_j)\rangle_\Omega - \langle \nabla p, (\nabla\psi_j)\phi\rangle_\Omega.\end{aligned}$$

If we approximate u in $W^{k,q}_w(\Omega)$ by a sequence $(u_j)_j$ of $C^\infty(\overline{\Omega})$-functions then we obtain from the unweighted case that the associated solution to the weak Neumann problem p_n is contained in $W^{k+1,s}(\Omega)$ for every $s \in (1, \infty)$; see [20] for the case $k = 1$, repeating the arguments given there shows the same for $k = 2$. Choose s such that

$$L^s(\Omega) \hookrightarrow L^q_w(\Omega).$$

Then $\psi_j p_n \in W^{k+1,q}_w(H_{\alpha_j})$. Moreover, by the properties of the Helmholtz decomposition stated in Section 4.4 one has $\|\nabla p\|_{q,w} \leq c\|u\|_{q,w}$. Thus by induction we may assume $\|\nabla p\|_{k-1,q,w} \leq c\|u\|_{k-1,q,w}$ and using the Lemmas A.2.1 and A.2.2 we obtain

$$\begin{aligned}\|\nabla(\psi_j p_n)\|_{k,q,w} \leq &\sup_{\phi\in\widehat{W}^{1,q'}_{w'}(H_{\alpha_j})} \frac{\langle \nabla(\psi_j p_n), \nabla\phi\rangle_\Omega}{\|\nabla\phi\|_{-k,q',w',0}} \\ \leq &\|p_n\nabla\psi_j\|_{k,q,w} + \|\psi_j u_n\|_{k,q,w} \\ &+ c \sup_{\phi\in\widehat{W}^{1,q'}_{w'}(H_{\alpha_j}), \operatorname{supp}\phi\subset B_r} \left(\|u_n\|_{k,q,w}\frac{\|\phi\|_{-k,q',w',0}}{\|\nabla\phi\|_{-k,q',w',0}} + \|\nabla p_n\|_{k-1,q,w}\frac{\|\phi\|_{1-k,q',w',0}}{\|\nabla\phi\|_{-k,q',w',0}}\right) \\ \leq &c\|u_n\|_{k,q,w},\end{aligned}$$

where the supremum is taken over all test functions ϕ with mean value 0. Summing over j we obtain $\|\nabla p_n\|_{k,q,w} \leq c\|u_n\|_{k,q,w}$. Since p_n converges to p in $W_w^{1,q}(\Omega)$ and since the convergence holds even in $W_w^{k,q}(\Omega)$ due to the above estimate, it follows

$$\|P_{q,w}u\|_{k,q,w} \leq \|u\|_{k,q,w} + c\|\nabla p\|_{k,q,w} \leq c\|u\|_{k,q,w}.$$

$\square$

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